CONTINUITY OF SRB MEASURE AND ENTROPY FOR BENEDICKS-CARLESON QUADRATIC MAPS

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Abstract. We consider the quadratic family of maps given by $f_a(x) = 1 - ax^2$ on $I = [-1, 1]$, for the Benedicks-Carleson parameters. On this positive Lebesgue measure set of parameters close to $a = 2$, f_a presents an exponential growth of the derivative along the orbit of the critical point and has an absolutely continuous Sinai-Ruelle-Bowen (SRB) invariant measure. We show that the volume of the set of points of I that at a given time fail to present an exponential growth of the derivative decays exponentially as time passes. We also show that the same applies to the volume of the set of points of I that are not slowly recurrent to the critical set. As a consequence we obtain continuous variation of the SRB measures and associated metric entropies with the parameter on the referred set. For this purpose we elaborate on the Benedicks-Carleson techniques in the phase space setting.

1. Introduction

Our object of study is the logistic family. Concerning the asymptotic behavior of orbits of points $x \in I = [-1, 1]$ we know that:

(1) The set of parameters H for which f_a has an attracting periodic orbit, is open and dense in $[0, 2]$.
(2) There is a positive Lebesgue measure set of parameters, close to the parameter value 2, for which f_a has no attracting periodic orbit and exhibits a chaotic behavior, in the sense of existence of an ergodic, f_a-invariant measure absolutely continuous with respect to the Lebesgue measure on $I = [-1, 1]$.
(3) There is also a well studied set of parameters where f_a is infinitely renormalizable.

The first result is a conjecture with long history, which was finally proved by Graczyk, Swiatek [GS97] and Lyubich [Ly97, Ly00]. The second one was studied in Jakobson's pioneer work [Ja81], in the work of P. Collet and J.P. Eckmann [CE83] and latter by Benedicks and Carleson in their celebrated papers [BC85, BC91], just to mention a few. For the third type of parameters we refer to [MS93] where an extensive treatment of the subject can be found.

Date: November 18, 2006.

2000 *Mathematics Subject Classification.* 37A35, 37C40, 37C75, 37D25, 37E05.

Key words and phrases. logistic family, Benedicks-Carleson parameters, SRB measures, entropy, non-uniform expansion, slow recurrence, large deviations.

Work supported by Fundação Calouste Gulbenkian.

The crucial role played by the orbit of the unique critical point $\xi_0 = 0$ on the determination of the dynamical behavior of f_a is remarkable. It is well known that if f_a has an attracting periodic orbit then $\xi_0 = 0$ belongs to its *basin of attraction*, which is the set of points $x \in I$ whose ω-limit set is the attracting periodic orbit. Also, the *basin of attraction* of the periodic orbit is an open and dense full Lebesgue measure subset of I. See [MS93], for instance.

Benedicks and Carleson [BC85, BC91] show the existence of a positive Lebesgue measure set of parameters $\mathcal{B}C_1$ for which there is exponential growth of the derivative of the orbit of the critical point ξ_0. This implies the non-existence of attracting periodic orbits and leads to a new proof of Jakobson's theorem.

In this work, we study the regularity in the variation of invariant measures and their metric entropy for small perturbations in the parameters. We are interested in investigating *statistical stability* of the system, that is, the persistence of its statistical properties for small modifications of the parameters. Alves and Viana [AV02] formalized the concept of *statistical stability* in terms of continuous variation of *physical measures* as a function of the governing law of the dynamical system.

By *physical measure* or *Sinai-Ruelle-Bowen (SRB) measure* we mean a Borel probability measure μ on I for which there is a positive Lebesgue measure set of points $x \in I$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\left(f_a^j(x)\right) = \int \varphi \, d\mu,$$

for any continuous function $\varphi : I \to \mathbb{R}$. The set of points $x \in I$ with this property is called the *basin* of μ. One should regard SRB measures as Borel probability measures that provide a fairly description of the statistical behavior of orbits, at least for a large set of points that constitute the *basin* of the SRB measure.

It is not difficult to conclude that if $a \in H$, and $\{p, f_a(p), \ldots, f_a^{k-1}(p)\}$ is the attracting periodic orbit then

$$\eta_a = \frac{1}{k} \sum_{i=0}^{k-1} \delta_{f_a^i(p)},$$

where δ_x is the Dirac probability measure at $x \in I$, is an SRB measure whose *basin* coincides with the *basin of attraction* of the periodic orbit. Moreover, the quadratic family is *statistically stable* for $a \in H$, i.e. the SRB measure η_a varies continuously with $a \in H$, in a weak sense (convergence of measures in the weak* topology).

The infinitely renormalizable quadratic maps also admit a SRB measure with the whole interval I for *basin*. In fact, any absolutely continuous f_a-invariant measure is SRB and describes (statistically speaking) the asymptotic behavior of almost all points, which is to say that its *basin* is I(see pp 348-352 [MS93]).

Benedicks and Young [BY92] proved that for each Benedicks-Carleson parameter $a \in \mathcal{B}C_1$, there is a unique, ergodic, f_a-invariant, absolutely continuous measure (with respect to Lebesgue measure on I) μ_a. These measures qualify as SRB measures by Birkhoff's ergodic theorem and their *basin* is the whole interval I.

Hence, it is a natural question to wonder if the Benedicks-Carleson quadratic maps are *statistically stable.*

In the subsequent sections it will be shown that the answer is in the affirmative. In fact, we will prove that the quadratic family is *statistically stable*, in strong sense, for $a \in \mathcal{B}C_1$. To be more precise, we will show that the densities of the SRB measures vary continuously, in L^1-norm, with the parameter $a \in \mathcal{B}C_1$. This result relates to those of Tsujii, Rychlik and Sorets. In [Ts96], Tsujii showed the continuity of SRB measures, in weak topology, on a positive Lebesgue measure set of parameters. Rychlik and Sorets [RS97], on the other hand, obtained the continuous variation of the SRB measures, in terms of convergence in L^1- norm, for Misiurewicz parameters, which form a subset of zero Lebesgue measure. We also would like to refer the work of Thunberg [Th01] who proved that on any full Lebesgue measure set of parameters there is no continuous variation of the SRB measure with the parameter.

With a view to studying the stability of the statistical behavior of the system in a broader perspective, we are also specially interested in the variation of entropy. Entropy is related to the unpredictability of the system. Topological entropy measures the complexity of a dynamical system in terms of the exponential growth rate of the number of orbits distinguishable over long time intervals, within a fixed small precision. Metric entropy with respect to an SRB measure, quantifies the average level of uncertainty every time we iterate, in terms of exponential growth rate of the number of statistically significant paths an orbit can follow.

It is known that topological entropy varies continuously with $a \in [0, 2]$ (see [MS93]). This is not the case with the metric entropy of SRB measures. We note that the metric entropy associated to η_a, with $a \in H$, is zero. H is an open and dense set which means we can find a sequence of parameters $(a_n)_{n\in\mathbb{N}}$, such that $a_n \in H$ and thus with zero metric entropy with respect to the SRB measure η_{a_n}, accumulating on $a \in \mathcal{B}C_1$, whose metric entropy with the absolutely continuous SRB measure, μ_a, is strictly positive.

However, we will show that the metric entropy of the absolutely continuous SRB measure μ_a varies continuously on the Benedicks-Carleson parameters, $a \in \mathcal{B}C_1$. We would like to stress that the continuous variation of the metric entropy is not a direct consequence of the continuous variation of the SRB measures and the entropy formula, because $\log(f'_a)$ is not continuous on the interval I.

1.1. **Motivation and main strategy.** The work developed by Alves and Viana on [AV02] led Alves [Al03] to obtain sufficient conditions for the strong statistical stability of certain classes of *non-uniformly expanding* maps with *slow recurrence to the critical set.* By *non-uniformly expanding*, we mean that for Lebesgue almost all points we have exponential growth of the derivative along their orbits. *Slow recurrence to the critical set* means, roughly speaking, that almost none of the points can have its orbit making frequent visits to very small vicinities of the critical set.

Alves, Oliveira and Tahzibi [AOT] determined abstract conditions for continuous variation of metric entropy with respect to SRB measures. They also obtained conditions for *non-uniformly expanding* maps with *slow recurrence to the critical set* to satisfy their initial abstract conditions.

In both cases, the conditions obtained for continuous variation of SRB measures and their metric entropy are tied with the volume decay of the *tail set*, which is the set of points that resist to satisfy either the *non-uniformly expanding* or the *slow recurrence to the critical set* conditions, up to a given time.

Consequently, our main objective is to show that on the Benedicks-Carleson set of parameter values, where we have exponential growth of the derivative along the orbit of the critical point $\xi_0 = 0$, the maps f_a are *non-uniformly expanding*, have *slow recurrence to the critical set*, and the volume of the *tail set* decays sufficiently fast. In fact, we will show that the volume of the tail set decays exponentially fast. Finally we apply the results on [Al03, AOT] to obtain the continuous variation of the SRB measures and their metric entropy inside the set of Benedicks-Carleson parameters $\mathcal{BC}_1$.

We also refer to the recent work [ACP06] from which we conclude, by the *non-uniformly expanding* character of these maps, that for almost every $x \in I$ and any y on a pre-orbit of x, one has an exponential growth of the derivative of y.

1.2. **Statement of results.** In what follows, we will only consider parameter values $a \in \mathcal{BC}_1$ that are Benedicks-Carleson parameters, in the sense that for those $a \in \mathcal{BC}_1$ we have exponential growth of the derivative of $f_a(\xi_0)$,

$$\left|\left(f_a^j\right)'(f_a(\xi_0))\right| \geq \mathrm{e}^{cj},\ \forall j \in \mathbb{N}, \tag{EG}$$

where $c \in \left[\frac{2}{3}, \log 2\right)$ is fixed, and the basic assumption is valid, namely

$$\left|f_a^j(\xi_0)\right| \geq \mathrm{e}^{-\alpha j},\ \forall j \in \mathbb{N}, \tag{BA}$$

where α is a small constant. Note that $\mathcal{BC}_1$ is a set of parameter values of positive Lebesgue measure, very close to $a = 2$. (See Theorem 1 of [BC91] or [Mo92] for a detailed version of its proof).

We say that f_a is *non-uniformly expanding* if there is a $d > 0$ such that for Lebesgue almost every point in $I = [-1, 1]$

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log\left|f_a'\left(f_a^i(x)\right)\right| > d, \tag{1.1}$$

while having *slow recurrence to the critical set* means that for every $\epsilon > 0$, there exists $\gamma > 0$ such that for Lebesgue almost every $x \in I$

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} -\log \mathrm{dist}_\gamma\left(f_a^j(x), 0\right) < \epsilon, \tag{1.2}$$

where

$$\mathrm{dist}_\gamma(x,y) = \begin{cases} |x-y| & \text{if } |x-y| \leq \gamma \\ 0 & \text{if } |x-y| > \gamma \end{cases}. \tag{1.3}$$

Observe that by (EG) it is obvious that ξ_0 satisfies (1.1) for all $a \in \mathcal{B}C_1$. However, with reference to condition (1.2) the matter is far more complicated and one has that ξ_0 satisfies it for Lebesgue almost all parameters $a \in \mathcal{B}C_1$. We provide a heuristic argument for the validity of the last statement on remark 8.2.

It is well known that the validity of (1.1) Lebesgue almost everywhere (a.e.) derives from the existence of an ergodic absolutely continuous invariant measure. Nevertheless we are also interested in knowing how fast does the volume of the points that resist to satisfy (1.1) up to n, decays to 0 as n goes to ∞. With this in mind, we define the *expansion time function*, first introduced on [ALP05]

$$\mathcal{E}^a(x) = \min\left\{ N \geq 1 : \frac{1}{n}\sum_{i=0}^{n-1} \log\left|f_a'\left(f_a^i(x)\right)\right| > d,\ \forall n \geq N \right\}, \tag{1.4}$$

which is defined and finite almost everywhere on I if (1.1) holds a.e.

Similarly, we define the *recurrence time function*, also introduced on [ALP05]

$$\mathcal{R}^a(x) = \min\left\{ N \geq 1 : \frac{1}{n}\sum_{j=0}^{n-1} -\log \operatorname{dist}_\gamma\left(f_a^j(x), 0\right) < \epsilon,\ \forall n \geq N \right\}, \tag{1.5}$$

which is defined and finite almost everywhere in I, as long as (1.2) holds a.e.

We are now able to define the *tail set*, at time $n \in \mathbb{N}$,

$$\Gamma_n^a = \{x \in I :\ \mathcal{E}^a(x) > n \text{ or } \mathcal{R}^a(x) > n\}, \tag{1.6}$$

which can be seen as the set of points that at time n have not reached a satisfactory exponential growth of the derivative or could not be sufficiently kept away from $\xi_0 = 0$.

First we study the volume contribution to the *tail set*, Γ_n^a, of the points where f_a fails to present *non-uniformly expanding* behavior. We claim that in fact, (1.1) a.e. holds to be true and the volume of the set of points whose derivative has not achieved a satisfactory exponential growth at time n, decays exponentially as n goes to ∞. In what follows λ denotes Lebesgue measure on $\mathbb{R}$.

Theorem A. *Assume that $a \in \mathcal{B}C_1$. Then f_a is non-uniformly expanding, which is to say that* (1.1) *holds for Lebesgue almost all points $x \in I$. Moreover, there are positive real numbers C_1 and τ_1 such that for all $n \in \mathbb{N}$:*

$$\lambda\{x \in I :\ \mathcal{E}^a(x) > n\} \leq C_1 e^{-\tau_1 n}.$$

Second, we study the volume contribution to Γ_n^a, of the points that fail to be *slowly recurrent* to ξ_0. We claim that (1.2) a.e. also holds true and the volume of the set of points that at time n, have been too close to the critical point, in mean, decays exponentially with n.

Theorem B. *Assume that $a \in \mathcal{B}C_1$. Then f_a has slow recurrence to the critical set, or in other words,* (1.2) *holds for Lebesgue almost all points $x \in I$. Moreover, there are positive real numbers C_2 and τ_2 such that for all $n \in \mathbb{N}$:*

$$\lambda\{x \in I :\ \mathcal{R}^a(x) > n\} \leq C_2 e^{-\tau_2 n}.$$

Remark 1.1. The constants d in (1.1), ϵ, γ in (1.2), c, α from (EG) and (BA) can be chosen uniformly on $\mathcal{B}C_1$. Moreover, the constants C_1, τ_1 given by theorem A and the constants C_2, τ_2 given by theorem B depend on the previous ones but are independent of the parameter $a \in \mathcal{B}C_1$. Thus, we may say that $\{f_a\}_{a\in\mathcal{B}C_1}$ is a *uniform family* in the sense considered in [Al03]. For a further discussion on this subject see section 9.

Remark 1.2. Both theorems easily imply that the volume of the *tail set* decays to 0 at least exponentially as n goes to ∞, i.e. for all $n \in \mathbb{N}$, $\lambda\left(\Gamma_n^a\right) \leq \text{const}\, \mathrm{e}^{-\tau n}$, for some $\tau > 0$ and const > 0.

The exponential volume decay of the *tail set* allows us to apply theorem A from [Al03] to obtain, in a strong sense, continuous variation of the ergodic invariant measures under small perturbations on the set of parameters. By strong sense we mean convergence of the densities of the ergodic invariant measures in the L^1 norm.

Corollary C. *Let μ_a be the SRB measure invariant for f_a. Then $\mathcal{B}C_1 \ni a \mapsto \frac{d\mu_a}{d\lambda}$ is continuous.*

Theorems A and B also make it possible to apply corollary C from [AOT] to get the continuous variation of metric entropy with the parameter.

Corollary D. *The entropy of the SRB measure invariant of f_a varies continuously with $a \in \mathcal{B}C_1$.*

Theorem A alone, also allows us to apply corollary 1.2 from [ACP06] to obtain backward contraction on every pre-orbit of Lebesgue almost every point.

Corollary E. *For Lebesgue almost every $x \in I$, there exists $C_x > 0$ and $b > 0$ such that $\left|(f_a^n)'(y)\right| > C_x e^{bn}$, for every $y \in f^{-n}(x)$ and for all $n \in \mathbb{N}$.*

2. Benedicks-Carleson techniques on phase space and notation

The first thing we need to establish is the meaning of "close to the critical set" and "distant from the critical set", for which we introduce the following neighborhoods of $\xi_0 = 0$:

$$U_m = \left(-\mathrm{e}^{-m}, \mathrm{e}^{-m}\right), \quad U_m^+ = U_{m-1}, \quad \text{for } m \in \mathbb{N}$$

and consider a large positive integer Δ that will indicate when closeness to the critical region is relevant. In fact, here and henceforth, we define $\delta = \mathrm{e}^{-\Delta}$.

We will use λ to refer to Lebesgue measure on $\mathbb{R}$, although, sometimes we will write $|\omega|$ as an abbreviation of $\lambda(\omega)$, for $\omega \subset \mathbb{R}$.

We follow [BC85, BC91] and proceed for each point $x \in I$ as was done in ξ_0, by splitting the orbit of x into *free periods*, *returns*, *bound periods*, which occur in this order. Before we explain these concepts we introduce the following notation for the orbit of the critical point, $\xi_n = f_a^n(0)$, for all $n \in \mathbb{N}_0$.

The *free periods* correspond to periods of time in which we are certain that the orbit never visits the vicinity $U_\Delta = (-\delta, \delta)$ of ξ_0. During these periods the orbit of x experiences an exponential growth of its derivative $|(f_a^n)'(x)|$, provided we are close enough to the parameter value 2. In fact, the following lemma gives a first approach to the set $\mathcal{B}C_1$ by stating that we may have an exponential growth rate $0 < c_0 < \log 2$ of the derivative of the orbit of x during free periods, for all $a \in [a_0, 2]$, where a_0 is chosen sufficiently close to 2.

Lemma 2.1. *For every* $0 < c_0 < \log 2$ *and* Δ *sufficiently large there exists* $1 < a_0(c_0, \Delta) < 2$ *such that for every* $x \in I$ *and* $a \in [a_0, 2]$ *one has:*

(1) *If* $x, f_a(x), \dots, f_a^{k-1}(x) \notin U_{\Delta+1}$ *then* $\left|(f_a^k)'(x)\right| \geq e^{-(\Delta+1)} e^{c_0 k}$*;*
(2) *If* $x, f_a(x), \dots, f_a^{k-1}(x) \notin U_{\Delta+1}$ *and* $f_a^k(x) \in U_\Delta^+$*, then* $\left|(f_a^k)'(x)\right| \geq e^{c_0 k}$*;*
(3) *If* $x, f_a(x), \dots, f_a^{k-1}(x) \notin U_{\Delta+1}$ *and* $f_a^k(x) \in U_1$*, then* $\left|(f_a^k)'(x)\right| \geq \frac{4}{5} e^{c_0 k}$.

The proof relies on the fact that $f_2(x) = 1 - 2x^2$ is conjugate to $1 - 2|x|$. So it is only a question of choosing a sufficiently close to 2 for f_a to inherit the expansive behavior of f_2. See [BC85] or [Al92, Mo92] for detailed versions. In what follows, we assume that a_0 is sufficiently close to 2 so that $c_0 \geq \frac{2}{3}$.

Due to this exponential expansion outside the critical region one can prove that, for almost every point $x \in I$, it is impossible to keep its orbit away from U_Δ. We have a *return* of the orbit of a point to the neighborhood of $\xi_0 = 0$ if for some $j \in \mathbb{N}$, $f_a^j(x) \in U_\Delta = (-\delta, \delta)$. So a free period ends with what we call a *free return.* There are two types of free returns: the *essential* and *inessential* ones. In order to distinguish each type we need a sequence $\mathcal{P}_0, \mathcal{P}_1, \dots$ of partitions of I into intervals. We begin by partitioning U_Δ in the following way:

$$I_m = \left[\mathrm{e}^{-(m+1)}, \mathrm{e}^{-m}\right), \qquad I_m^+ = \left[\mathrm{e}^{-(m+1)}, \mathrm{e}^{-(m-1)}\right) \qquad, \text{for } m \geq \Delta,$$
$$I_m = \left(-\mathrm{e}^{-m}, -\mathrm{e}^{-(m+1)}\right], I_m^+ = \left(-\mathrm{e}^{-(m-1)}, -\mathrm{e}^{-(m+1)}\right], \text{for } m \leq -\Delta.$$

We say that the return had a *depth* of $\mu \in \mathbb{N}$ if $\mu = [-\log \operatorname{dist}_\delta(f_a^j(x), 0)]$, which is equivalent to saying that $f_a^j(x) \in I_{\pm\mu}$.

Next we subdivide each I_m, $m \geq \Delta$ into m^2 pieces of the same length in order to obtain bounded distortion on each member of the partition. For each $m \geq \Delta - 1$ and $k = 1, \dots, m^2$, we introduce the following notation

$$I_{m,k} = \left[\mathrm{e}^{-m} - k\frac{\lambda(I_m)}{m^2}, \mathrm{e}^{-m} - (k-1)\frac{\lambda(I_m)}{m^2}\right)$$
$$I_{-m,k} = -I_{m,k}\,, \qquad I_{m,k}^+ = I_{m_1,k_1} \cup I_{m,k} \cup I_{m_2,k_2},$$

where I_{m_1,k_1} and I_{m_2,k_2} are the adjacent intervals of $I_{m,k}$.

The sequence of partitions will be built in full detail on section 4 but we note the following:

For Lebesgue almost every $x \in I$, $\{x\} = \cap_{n \geq 0} \omega_n(x)$, where $\omega_n(x)$ is the element of $\mathcal{P}_n$ containing x. For such x there is a sequence $t_1, t_2, \dots$ corresponding to the instants when the orbit of x experiences an essential return, which means $I_{m,k} \subset f_a^{t_i}(\omega_{t_i}(x)) \subset I_{m,k}^+$ for some $|m| \geq \Delta$ and $1 \leq k \leq m^2$. In contrast we say that v is a free return time for x of inessential type if $f_a^v(\omega_v(x)) \subset I_{m,k}^+$, for some $|m| \geq \Delta$ and $1 \leq k \leq m^2$, but $f_a^v(\omega_v(x))$ is not large enough to contain an interval $I_{m,k}$ for some $|m| \geq \Delta$ and $1 \leq k \leq m^2$.

Now let us see some consequences of the returns. Since

$$|(f_a^n)'(x)| = \prod_{j=1}^{n} \left|2af_a^j(x)\right|,$$

the returns introduce some small factors in the derivative of the orbit of x. Also if we define for a point $x \in I$ and $n \in \mathbb{N}$,

$$T_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} -\log \operatorname{dist}_\gamma \left(f_a^j(x), 0\right), \tag{2.1}$$

where $\gamma = \mathrm{e}^{-\Theta}$ is the same of condition (1.2) and $\operatorname{dist}_\gamma$ is given by (1.3). We note that the only points of the orbit of x that contribute to the sum in (2.1) are those considered to be *deep* returns with depth above the threshold $\Theta \geq \Delta$ which is to be determined later. To compensate for the loss in the expansion of the derivative, we will show that a property very similar to (BA) holds for the orbit of $x \in I$ which can be seen as follows: we allow the orbit of x to get close to ξ_0 but we put some restraints on the velocity of possible accumulation on ξ_0. This will be the basis of the proof of theorem A. As for the proof of theorem B the strategy will be of different kind, it will be based on a statistical analysis of the depth of the returns, specially of the essential returns, which, fortunately, are very unlikely to reach large depths.

Finally, we are lead to the notion of *bound period* that follows a return during which the orbit of x is bounded to the orbit of ξ_0, or in other words: if at a return the orbit of x falls in a tight vicinity of the critical point we expect it to shadow the early iterates of ξ_0 at least for some period of time.

Let $\beta > 0$ be a small number such that $\beta > \alpha$; for example, take $10^{-2} > \beta = 2\alpha$.

Definition 2.2. Suppose $x \in U_m^+$. Let $p(x)$ be the largest p such that the following binding condition holds:

$$\left|f_a^j(x) - \xi_j(a)\right| \leq \mathrm{e}^{-\beta j}, \quad \text{for all } i = 1, \dots, p-1 \tag{BC}$$

The time interval $1, \dots, p(x) - 1$ is called the *bound period* for x.

If $p(m)$ is the largest p such that (BC) holds for all $x \in I_m^+$, which is the same to define

$$p(m) = \min_{x \in I_m^+} p(m, x),$$

then the time interval $1, \dots, p(m) - 1$ is called the *bound period* for I_m^+.

One expects that the deeper is the return, the longer is its associated bound period. Next lemma confirms this, in particular.

Lemma 2.3. *If Δ is sufficiently large, then for each $|m| \geq \Delta$, $p(m)$ has the following properties:*

(1) *There is a constant $B_1 = B_1(\beta - \alpha)$ such that $\forall y \in f_a\left(U_{|m|-1}\right)$*

$$\frac{1}{B_1} \leq \left|\frac{(f_a^j)'(y)}{(f_a^j)'(\xi_1)}\right| \leq B_1, \quad \textit{for } j = 0, 1 \dots, p(m) - 1;$$

(2) $\frac{2}{3}|m| < p(m) < 3|m|$;
(3) $\left|(f_a^p)'(x)\right| \geq e^{(1-4\beta)|m|}$, *for* $x \in I_m^+$ *and* $p = p(m)$.

The proof of this lemma depends heavily on the conditions (EG) and (BA). It can be found in [Al92, Mo92]. (See [BC85] for a similar version of the lemma but with sub-exponential estimates).

We call the attention to the fact that after the bound period not only have we recovered from the loss on the growth of the derivative caused by the return that originated the bound period, but we even have some exponential gain.

Also note that nothing prevents the orbit of a point x from entering in U_Δ during a bound period. These instants are called the *bound return* times.

Hence, we may speak of three types of returns: *essential*, *inessential* and *bound*. The essential returns are the ones that will play a prominent role in the reasoning. Let, as before, the sequence $t_1, t_2, \ldots$ denote the instants corresponding to essential returns of the orbit of x. When $n \in \mathbb{N}$ is given, we can define s_n to be the number of essential returns of the orbit of x, occurring up to n. We denote by $sd_n(x)$ the number of those essential returns occurring up to n that correspond to deep essential returns of the orbit of x with return depths above the threshold $\Theta \geq \Delta$. Let η_i denote the depth of the i-th essential return. Each t_i may be followed by bounded returns at times $u_{i,j}$, $j = 1, \ldots, u$ and these can be followed by inessential returns at times $v_{i,j}$, $j = 1, \ldots, v$. We will write $\eta_{i,j}$ to denote the depth of the inessential return correspondent to $v_{i,j}$. Note that each $v_{i,j}$ has a bound evolution where new bound returns may occur and, although we refer to these returns later, it is not necessary to introduce here a notation for them. Sometimes, for the sake of simplicity, it is convenient not to distinguish between essential and inessential returns, so we introduce the notation $z_1 < z_2 < \ldots$ for the instants of occurrence of free returns of the orbit of x.

We call attention to the fact that t_i, for example, depends on the point $x \in I$ considered- $t_i(x)$ corresponds to the i-th instant of essential return of the orbit of x. So, t_i, s_n, η_i, $u_{i,j}$, $v_{i,j}$, $\eta_{i,j}$ and z_i, should be regarded as functions of the point $x \in I$.

The sequence of partitions $\mathcal{P}_n$ of the set I will be such that all $x \in \omega \in \mathcal{P}_n$ have the same return times and return depths up to n. In fact, if, for example, $t_i(x) \leq n$ for some $x \in \omega \in \mathcal{P}_n$, then t_i and η_i are constant on ω. The same applies to the other above mentioned functions of x. The construction of the partition will also guarantee that f_a has bounded distortion on each component which will be shown to be of extreme importance.

3. Insight into the reasoning

We are now in condition to sketch the proofs of theorems A and B. The following two basic ideas are determinant for both the proofs.

(I) Not only the depth of the inessential and bound returns is smaller than the depth of the essential return preceding them (as we will show in lemmas 5.1 and 5.2) but also the total sum of the depths of bounded and inessential returns is less than a quantity proportional to the depth of the essential return preceding them, as we will show in propositions 5.4 and 5.5.

(II) The chances of occurring a very deep essential return are very small, in fact, they are less than $e^{-\tau\rho}$, where $\tau > 0$ is constant and ρ is the depth in question. See proposition 6.2 and corollary 6.3.

The first one derives from (BA), (EG) and other properties of the critical orbit, while the main ingredient of the proof of the second is the bounded distortion on each element of the partition.

In order to prove theorem A, we define the following sets for a sufficiently large n.

$$E_1(n) = \left\{x \in I : \exists i \in \{1, \dots, n\},\ |f_a^i(x)| < e^{-\alpha n}\right\}. \tag{3.1}$$

Next, we will see that if $x \in I - E_1(n)$ then $\left|(f_a^n)'(x)\right| > e^{dn}$, for some $d = d(\alpha, \beta) > 0$.

Let us fix a large n. Assume that z_i, $i = 1, \dots, \gamma$ are the instants of return of the orbit of x, either essential or inessential. Let p_i denote the length of the bound period associated with the return z_i. We set $z_0 = 0$, whether $x \in U_\Delta$ or not; $p_0 = 0$ if $x \notin U_\Delta$ and as usual if not. We define $q_i = z_{i+1} - (z_i + p_i)$, for $i = 0, 1, \dots, \gamma - 1$ and

$$q_\gamma = \begin{cases} 0 & \text{if } n < z_\gamma + p_\gamma \\ n - (z_\gamma + p_\gamma) & \text{if } n \geq z_\gamma + p_\gamma \end{cases}.$$

Finally, let

$$d = \min\left\{c, \frac{1-4\beta}{3}\right\} - 2\alpha = \frac{1-4\beta}{3} - 2\alpha. \tag{3.2}$$

If $n \geq z_\gamma + p_\gamma$ then

$$|(f_a^n)'(x)| = \prod_{i=0}^{\gamma} \left|(f_a^{q_i})'(f_a^{z_i+p_i}(x))\right| |(f_a^{p_i})'(f_a^{z_i}(x))|.$$

Using lemmas 2.1 and 2.3, we have

$$|(f_a^n)'(x)| \geq e^{-\Delta+1} e^{c_0 \sum_{i=0}^{\gamma} q_i} e^{\frac{1-4\beta}{3}\sum_{i=0}^{\gamma} p_i} \geq e^{-\Delta+1} e^{dn} e^{2\alpha n} \geq e^{dn}, \tag{3.3}$$

for n large enough.

If $n < z_\gamma + p_\gamma$ then

$$|(f_a^n)'(x)| = |f_a'(f_a^{z_\gamma}(x))| \left|(f_a^{n-(z_\gamma+1)})'(f_a^{z_\gamma+1}(x))\right| \prod_{i=0}^{\gamma-1} \left|(f_a^{q_i})'(f_a^{z_i+p_i}(x))\right| |(f_a^{p_i})'(f_a^{z_i}(x))|.$$

Now, by lemmas 2.1 and 2.3 together with the assumption that $x \in I - E_1(n)$, for n large enough we have

$$\begin{aligned} |(f_a^n)'(x)| &\geq |f_a'(f_a^{z_\gamma}(x))| \tfrac{1}{B_1} \left|(f_a^{n-(z_\gamma+1)})'(1)\right| e^{c_0 \sum_{i=0}^{\gamma-1} q_i} e^{\frac{1-4\beta}{3}\sum_{i=0}^{\gamma-1} p_i} \\ &\geq e^{-\alpha n} \tfrac{1}{B_1} e^{c_0(n-(z_\gamma+1))} e^{c_0 \sum_{i=0}^{\gamma-1} q_i} e^{\frac{1-4\beta}{3}\sum_{i=0}^{\gamma-1} p_i} \\ &\geq e^{-\alpha n - \log B_1} e^{(d+2\alpha)(n-1)} \\ &\geq e^{-2\alpha n} e^{dn} e^{2\alpha n} \\ &\geq e^{dn}. \end{aligned} \tag{3.4}$$

Using (I) and (II) we will show that

$$\lambda\left(E_1(n)\right) \leq \mathrm{e}^{-\tau_1 n}, \tag{3.5}$$

for a constant $\tau_1(\alpha,\beta) > 0$ and for all $n \geq N_1^*(\Delta,\tau_1)$. We consider $N_1(\Delta,\alpha,B_1,d,N_1^*)$ such that for all $n \geq N_1$ estimates (3.3), (3.4) and (3.5) hold. Hence for every $n \geq N_1$ we have that $|(f_a^n)'(x)| \geq \mathrm{e}^{dn}$, except for a set $E_1(n)$ of points $x \in I$ satisfying (3.5).

We take $E_1 = \bigcap_{k\geq N_1}\bigcup_{n\geq k} E_1(n)$. Since $\forall k \geq N_1$

$$\sum_{n\geq k}\lambda\left(E_1(n)\right) \leq \mathrm{const}\,\mathrm{e}^{-\tau_1 k},$$

we have by the Borel Cantelli lemma that $\lambda(E_1) = 0$. Thus on the full Lebesgue measure set $I - E_1$ we have that (1.1) holds. We note that $\{x \in I:\ \mathcal{E}^a(x) > k\} \subset \bigcup_{n\geq k} E_1(n)$, where $\mathcal{E}^a$ is defined in (1.4). So for $k \geq N_1$

$$\lambda\left(\{x \in I:\ \mathcal{E}^a(x) > k\}\right) \leq \mathrm{const}\,\mathrm{e}^{-\tau_1 k}.$$

At this point we just have to compute an adequate $C_1 = C_1(N_1) > 0$ such that

$$\lambda\left(\{x \in I:\ \mathcal{E}^a(x) > n\}\right) \leq C_1\mathrm{e}^{-\tau_1 n}, \tag{3.6}$$

for all $n \in N$.

For the proof of theorem B, we define for $n \in \mathbb{N}$ the sets:

$$E_2(n) = \{x \in I:\ T_n(x) > \epsilon\}. \tag{3.7}$$

Note that it is the depth of the deep returns that counts for the sum on $T_n(x)$. Taking note of the basic idea (I), in order to obtain a bound for T_n one only needs to take into consideration the deep essential returns.

Thus if we define

$$F_n(x) = \sum_{i=1}^{sd_n}\eta_i, \tag{3.8}$$

where sd_n is the number of essential returns with depths above Θ that occur up to n and η_i their respective depths, we have $T_n(x) \leq \frac{C_5}{n}F_n(x)$, from which we conclude that

$$\lambda(E_2(n)) \leq \lambda\left\{x : F_n(x) > \frac{\epsilon n}{C_5}\right\}.$$

Fact (II) and a large deviation argument allow us to obtain for $n \geq N_2(\Theta)$

$$\lambda\left\{x : F_n(x) > \frac{\epsilon n}{C_5}\right\} \leq \mathrm{const}\,\mathrm{e}^{-\tau_2 n}$$

where $\tau_2 = \tau_2(\epsilon,\Theta) > 0$ is constant, which implies for $k \geq N_2$

$$\sum_{n\geq k}\lambda\left(E_2(n)\right) \leq \mathrm{const}\,\mathrm{e}^{-\tau_2 k}.$$

Consequently, applying Borel Cantelli's lemma, we get $\lambda(E_2)=0$, where $E_2=\cap_{k\geq 1}\cup_{n\geq k} E_2(n)$ and finally conclude that (1.2) holds on the full Lebesgue measure set $I-E_2$. Observe that $\{x\in I:\ \mathcal{R}^a(x)>k\}\subset\bigcup_{n\geq k}E_2(n)$, and thus, for all $n\in\mathbb{N}$,

$$\lambda\left(\{x\in I:\ \mathcal{R}^a(x)>n\}\right)\leq C_2\mathrm{e}^{-\tau_2 n},$$

where $C_2=C_2(N_2,\tau_2)>0$ is constant. Recall that $\mathcal{R}^a$ is defined in (1.5).

At this point we would like to bring the reader's attention to the fact that most proofs and lemmas that follow are standard, in the sense that they are very resemblant to the ones on [Al92, BC85, BC91, BY92, Mo92] (just to cite a few), that deal with the same subject. Nevertheless, we could not find the right version for our needs, either because in some cases they refer to sub-exponential estimates when we want exponential estimates or because the partition is built on the space of parameters instead of the set I, as we wish. Hence, we decided for the sake of completeness to include them in this work.

4. Construction of the partition and bounded distortion

We are going to build inductively a sequence of partitions $\mathcal{P}_0,\mathcal{P}_1,\dots$ of I (modulo a zero Lebesgue measure set) into intervals. We will also define inductively the sets $R_n(\omega)=\left\{z_1,\dots,z_{\gamma(n)}\right\}$ which is the set of the return times of $\omega\in\mathcal{P}_n$ up to n and a set $Q_n(\omega)=\left\{(m_1,k_1),\dots,(m_{\gamma(n)},k_{\gamma(n)})\right\}$, which records the indices of the intervals such that $f_a^{z_i}(\omega)\subset I^+_{m_i,k_i}$, $i=1,\dots,z_{\gamma(n)}$.

Along with the construction of the partition, we will show, inductively, that for all $n\in\mathbb{N}_0$

$$\forall\omega\in\mathcal{P}_n\quad f_a^{n+1}|_\omega \text{ is a diffeomorphism,}\tag{4.1}$$

which is vital for the construction itself.

For $n=0$ we define

$$\mathcal{P}_0=\{[-1,-\delta],[\delta,1]\}\cup\left\{I_{m,k}:|m|\geq\Delta,\ 1\leq k\leq m^2\right\}.$$

It is obvious that $\mathcal{P}_0$ satisfies (4.1). We set $R_0\left([-1,-\delta]\right)=R_0\left([\delta,1]\right)=\emptyset$ and $R_0(I_{m,k})=\{0\}$.

Assume that $\mathcal{P}_{n-1}$ is defined, satisfies (4.1), and R_{n-1}, Q_{n-1} are also defined on each element of $\mathcal{P}_{n-1}$. We fix an interval $\omega\in\mathcal{P}_{n-1}$. We have three possible situations:

(1) If $R_{n-1}(\omega)\neq\emptyset$ and $n<z_{\gamma(n-1)}+p(m_{\gamma(n-1)})$ then we say that n is a *bound time* for ω, put $\omega\in\mathcal{P}_n$ and set $R_n(\omega)=R_{n-1}(\omega)$, $Q_n(\omega)=Q_{n-1}(\omega)$.

(2) If $R_{n-1}(\omega)=\emptyset$ or $n\geq z_{\gamma(n-1)}+p(m_{\gamma(n-1)})$), and $f_a^n(\omega)\cap U_\Delta\subset I_{\Delta,1}\cup I_{-\Delta,1}$, then we say that n is a *free time* for ω, put $\omega\in\mathcal{P}_n$ and set $R_n(\omega)=R_{n-1}(\omega)$, $Q_n(\omega)=Q_{n-1}(\omega)$.

(3) If the above two conditions do not hold we say that ω has a *free return situation* at time n. We have to consider two cases:
   (a) $f_a^n(\omega)$ does not cover completely an interval $I_{m,k}$, with $|m|\geq\Delta$ and $k=1,\dots,m^2$. Because f_a^n is continuous and ω is an interval, $f_a^n(\omega)$ is also an interval and thus is contained in some $I^+_{m,k}$, for a certain $|m|\geq\Delta$ and $k=1,\dots,m^2$, which is called the *host interval* of the return. We say that n is an

inessential return time for ω, put $\omega \in \mathcal{P}_n$ and set $R_n(\omega) = R_{n-1}(\omega) \cup \{n\}$, $Q_n(\omega) = Q_{n-1}(\omega) \cup \{(m,k)\}$.

(b) $f_a^n(\omega)$ contains at least an interval $I_{m,k}$, with $|m| \geq \Delta$ and $k = 1, \dots, m^2$, in which case we say that ω has an *essential return situation* at time n. Then we consider the sets

$$\begin{aligned} \omega'_{m,k} &= f_a^{-n}(I_{m,k}) \cap \omega \quad \text{ for } |m| \geq \Delta \\ \omega'_+ &= f_a^{-n}([\delta, 1]) \cap \omega \\ \omega'_- &= f_a^{-n}([-1, -\delta]) \cap \omega \end{aligned}$$

and if we denote by $\mathcal{A}$ the set of indices (m,k) such that $\omega'_{m,k} \neq \emptyset$ we have

$$\omega - \left\{f_a^{-n}(0)\right\} = \bigcup_{(m,k)\in\mathcal{A}} \omega'_{m,k}. \tag{4.2}$$

By the induction hypothesis $f_a^n|_\omega$ is a diffeomorphism and then each $\omega'_{m,k}$ is an interval. Moreover $f_a^n(\omega'_{m,k})$ covers the whole $I_{m,k}$ except eventually for the two end intervals. When $f_a^n(\omega'_{m,k})$ does not cover $I_{m,k}$ entirely, we join it with its adjacent interval in (4.2). We also proceed likewise when $f_a^n(\omega'_+)$ does not cover $I_{\Delta-1,(\Delta-1)^2}$ or $f_a^n(\omega'_-)$ does not contain the whole interval $I_{1-\Delta,(\Delta-1)^2}$. In this way we get a new decomposition of $\omega - \{f_a^{-n}(0)\}$ into intervals $\omega_{m,k}$ such that

$$I_{m,k} \subset f_a^n(\omega_{m,k}) \subset I^+_{m,k},$$

when $|m| \geq \Delta$.

We define $\mathcal{P}_n$, by putting $\omega_{m,k} \in \mathcal{P}_n$ for all indices (m,k) such that $\omega_{m,k} \neq \emptyset$, with $|m| \geq \Delta$, which results in a refinement of $\mathcal{P}_{n-1}$ at ω. We set $R_n(\omega_{m,k}) = R_{n-1}(\omega) \cup \{n\}$ and n is called an *essential return time* for $\omega_{m,k}$. The interval $I^+_{m,k}$ is called the *host interval* of $\omega_{m,k}$ and $Q_n(\omega_{m,k}) = Q_n(\omega) \cup \{(m,k)\}$.

In the case when the set ω_+ is not empty we say that n is an *escape time* or *escape situation* for ω_+ and $R_n(\omega_+) = R_{n-1}(\omega)$, $Q_n(\omega_+) = Q_{n-1}(\omega)$. We proceed likewise for ω_-. We also refer to ω_+ or ω_- as *escaping components*. Note that the points in escaping components are in free period.

To end the construction we need to verify that (4.1) holds for $\mathcal{P}_n$. Since for any interval $J \subset I$

$$\left.\begin{array}{l} f_a^n|_J \text{ is a diffeomorphism} \\ 0 \notin f_a^n(J) \end{array}\right\} \Rightarrow f_a^{n+1}|_J \text{ is a diffeomorphism},$$

all we are left to prove is that $0 \notin f_a^n(\omega)$ for all $\omega \in \mathcal{P}_n$. So take $\omega \in \mathcal{P}_n$. If n is a free time for ω then we have nothing to prove. If n is a return for ω, either essential or inessential, we have by construction that $f_a^n(\omega) \subset I^+_{m,k}$ for some $|m| \geq \Delta$, $k = 1, \dots, m^2$ and thus $0 \notin f_a^n(\omega)$. If n is a bound time for ω then by definition of bound period and (BA) we have for all $x \in \omega$

$$\begin{aligned} |f_a^n(x)| &\geq \left|f_a^{n-z_{\gamma(n-1)}}(0)\right| - \left|f_a^n(x) - f_a^{n-z_{\gamma(n-1)}}(0)\right| \\ &\geq \mathrm{e}^{-\alpha(n-z_{\gamma(n-1)})} - \mathrm{e}^{-\beta(n-z_{\gamma(n-1)})} \\ &\geq \mathrm{e}^{-\alpha(n-z_{\gamma(n-1)})}\left(1-\mathrm{e}^{-(\beta-\alpha)(n-z_{\gamma(n-1)})}\right) \\ &> 0 \quad \text{since } \beta-\alpha>0. \end{aligned}$$

Now we will obtain estimates of the length of $|f_a^n(\omega)|$.

Lemma 4.1. *Suppose that z is a return time for $\omega \in \mathcal{P}_{n-1}$, with host interval $I_{m,k}^+$. Let $p = p(m)$ denote the length of its bound period. Then*

1. *Assuming that $z^* \leq n-1$ is the next return time for ω (either essential or inessential) and defining $q = z^* - (z+p)$ we have, for a sufficiently large Δ, $\left|f_a^{z^*}(\omega)\right| \geq e^{c_0 q} e^{(1-4\beta)|m|} |f_a^z(\omega)| \geq 2\,|f_a^z(\omega)|$.*
2. *If z is the last return time of ω up to $n-1$ and n is either a free time for ω or a return situation for ω, then putting $q = n-(z+p)$ we have, for a sufficiently large Δ,*
   (a) $|f_a^n(\omega)| \geq e^{c_0 q-(\Delta+1)} e^{(1-4\beta)|m|} |f_a^z(\omega)|$
   (b) $|f_a^n(\omega)| \geq e^{c_0 q-(\Delta+1)} e^{-5\beta|m|}$ *if z is an essential return.*
3. *If z is the last return time of ω up to $n-1$, n is a return situation for ω and $f_a^n(\omega) \subset U_1$, then putting $q = n-(z+p)$ we have, for a sufficiently large Δ,*
   (a) $|f_a^n(\omega)| \geq e^{c_0 q} e^{(1-5\beta)|m|} |f_a^z(\omega)| \geq 2\,|f_a^z(\omega)|$;
   (b) $|f_a^n(\omega)| \geq e^{c_0 q} e^{-5\beta|m|}$ *if z is an essential return.*

Proof. By the mean value theorem, for some $\zeta \in \omega$,

$$|f_a^n(\omega)| \geq \left|\left(f_a^{n-z}\right)'(f_a^z(\zeta))\right| |f_a^z(\omega)|.$$

Using lemma 2.1 part 2 and lemma 2.3 part 3 we get

$$\begin{aligned} |f_a^n(\omega)| &\geq \left|(f_a^q)'\left(f_a^{z+p}(\zeta)\right)\right| \left|(f_a^p)'(f_a^z(\zeta))\right| |f_a^z(\omega)| \\ &\geq \tfrac{4}{5}\mathrm{e}^{c_0 q}\mathrm{e}^{(1-4\beta)|m|}\,|f_a^z(\omega)| \\ &\geq \tfrac{4}{5}\mathrm{e}^{\beta|m|}\mathrm{e}^{c_0 q}\mathrm{e}^{(1-5\beta)|m|}\,|f_a^z(\omega)| \\ &\geq 2\mathrm{e}^{c_0 q}\mathrm{e}^{(1-5\beta)|m|}\,|f_a^z(\omega)|, \end{aligned}$$

if Δ is sufficiently large in order to have $\frac{4}{5}\mathrm{e}^{\beta|m|} \geq 2$.

Note that part 3a is proved. To demonstrate part 1 it is only a matter of using lemma 2.1 part 2 instead of 3, while for proving part 2a one has to use lemma 2.1 part 1 instead.

To obtain 3b observe that because z is an essential return time $I_{m,k} \subset f_a^z(\omega)$ which implies $\lambda(f_a^z(\omega)) \geq \frac{e^{-|m|}}{2m^2}$ and so

$$\begin{aligned}|f_a^n(\omega)| &\geq \tfrac{4}{5} e^{\beta|m|} e^{c_0 q} e^{(1-5\beta)|m|} \, |f_a^z(\omega)| \\ &\geq e^{c_0 q} e^{(1-5\beta)|m|} e^{-|m|} \tfrac{2e^{\beta|m|}}{5m^2} \\ &\geq e^{c_0 q} e^{-5\beta|m|},\end{aligned}$$

if Δ is large enough.

The same argument can easily be applied to obtain part 2b. □

The next lemma asserts that an escaping component returns considerably large in the return situation immediately after the escaping time, which means in particular that it will be an essential return situation.

Lemma 4.2. *Suppose that $\omega \in \mathcal{P}_t$ is an escape component. Then in the next return situation t_1 for ω we have that*

$$\left|f_a^{t_1}(\omega)\right| \geq e^{-\beta\Delta}.$$

Proof. Since ω is an escaping component at time t it follows that

$$f_a^t(\omega) \supset I_{m,m^2}, \text{ with } |m| = \Delta - 1$$

and so there exists $x_* \in \omega$ such that $|f_a^t(x_*)| = e^{-\Delta}$. Therefore $f_a^{t+1}(x^*) = 1 - ae^{-2\Delta} \geq 1 - 2e^{-2\Delta}$. Thus, if $t_1 = t+1$ the result would follow easily.

Now suppose that $t_1 \geq t+2$. Writing

$$f_a^{t+2}(x_*) = f_2\left(f_a^{t+1}(x_*)\right) + f_a\left(f_a^{t+1}(x_*)\right) - f_2\left(f_a^{t+1}(x_*)\right)$$

and taking into account that $f_2\left(f_a^{t+1}(x_*)\right) \leq f_2(1-2e^{-2\Delta})$, $f_a(y) - f_2(y) \leq 2 - a$, $\forall y \in I$, it follows that

$$f_a^{t+2}(x_*) \leq -1 + 4.2e^{-2\Delta},$$

if we choose a_0 sufficiently close to 2 such that

$$(2 - a_0) \leq 8e^{-4\Delta}. \tag{4.3}$$

By induction, using the same argument we can state that for $k \geq 2$, providing that $-1 + 4^{k-2}2e^{-2\Delta} \leq 1$ (this is to ensure that we are inside the domain I), we have

$$f_a^{t+k}(x_*) \leq -1 + 4^{k-1}2e^{-2\Delta}.$$

Therefore if $-1 + 4^{t_1-t-1}2e^{-2\Delta} \leq -\frac{1}{2}$ then $f_a^{t_1}(x_*) \leq -\frac{1}{2}$ and so $|f_a^{t_1}(\omega)| \geq e^{-\beta\Delta}$, providing Δ is large enough.

In order to complete the proof it remains to consider the case when $-1 + 4^{t_1-t-1}2e^{-2\Delta} > -\frac{1}{2}$. Under this condition we have that

$$2^{t_1-t} \geq e^{\Delta}. \tag{4.4}$$

First we note that we can assume $f_a^t(\omega) \subset U_1$ otherwise we have the conclusion immediately.

Now, we know that there is $x \in \omega$ such that

$$\begin{aligned}\left|f_a^{t_1}(\omega)\right| &\geq \left|\left(f_a^{t_1-t}\right)'\left(f_a^t(x)\right)\right| \left|f_a^t(\omega)\right| \\ &\geq \frac{\left|(h^{-1})'\left(f_a^t(x)\right)\right|}{\left|(h^{-1})'\left(f_a^{t_1}(x)\right)\right|} \left|\left(g_a^{t_1-t}\right)'\left(h^{-1}(f_a^t(x))\right)\right| \frac{\mathrm{e}^{-\Delta}}{(\Delta-1)^2},\end{aligned}$$

where $h : [-1,1] \to [-1,1]$ is the homeomorphism that conjugates $f_2(x)$ to the tent map $1-2|x|$ and $g_a = h^{-1} \circ f_a \circ h$.

Using lemma 3.1 from [Mo92] it follows that

$$\left|f_a^{t_1}(\omega)\right| \geq L \left[2 - \tfrac{3\pi}{\delta^3}(2-a)\right]^{t_1-t} \frac{\mathrm{e}^{-\Delta}}{(\Delta-1)^2},$$

with

$$L = \sqrt{\frac{1-\left(f_a^{t_1}(x)\right)^2}{1-\left(f_a^t(x)\right)^2}}.$$

Since $f_a^{t_1}(\omega) \subset U_1$,

$$\begin{aligned}\left|f_a^{t_1}(\omega)\right| &\geq \sqrt{1-\mathrm{e}^{-2}} \left[2 - \tfrac{3\pi}{\delta^3}(2-a)\right]^{t_1-t} \frac{\mathrm{e}^{-\Delta}}{(\Delta-1)^2} \\ &\geq \tfrac{4}{5} \left[2 - \tfrac{3\pi}{\delta^3}(2-a)\right]^{t_1-t} \frac{\mathrm{e}^{-\Delta}}{(\Delta-1)^2}\end{aligned}$$

Now, we remark that our choice of a_0 can provide that

$$\left[2 - \tfrac{3\pi}{\delta^3}(2-a)\right] \geq \mathrm{e}^{c_0} \tag{4.5}$$

and then since $|f_a^{t_1}(\omega)| \leq 2$, it follows that

$$\mathrm{e}^{c_0(t_1-t)} \leq \tfrac{5}{2}\mathrm{e}^{\Delta}(\Delta-1)^2,$$

which implies, for Δ large that $t_1 - t \leq 2\Delta$.

Again restraining a_0 in such a way that

$$\left[2 - \tfrac{3\pi}{\delta^3}(2-a)\right]^{2\Delta} \geq 2^{2\Delta-1} \tag{4.6}$$

we have

$$\left|f_a^{t_1}(\omega)\right| \geq \tfrac{2}{5} 2^{t_1-t} \frac{\mathrm{e}^{-\Delta}}{(\Delta-1)^2}.$$

Taking into account 4.4 we have

$$\left|f_a^{t_1}(\omega)\right| \geq \frac{2}{5(\Delta-1)^2} \geq \mathrm{e}^{-\beta\Delta},$$

for Δ large enough. □

Lemma 4.3 (Bounded Distortion). *For some $n \in \mathbb{N}$ let $\omega \in \mathcal{P}_{n-1}$ be such that $f_a^n(\omega) \subset U_1$. Then there is a constant $C(\beta - \alpha)$ such that for every $x, y \in \omega$*

$$\frac{\left|(f_a^n)'(x)\right|}{\left|(f_a^n)'(y)\right|} \leq C$$

Proof. Let $R_{n-1}(\omega) = \{z_1, \ldots, z_\gamma\}$ and $Q_{n-1}(\omega) = \{(m_1, k_1), \ldots, (m_\gamma, k_\gamma)\}$, be, respectively, the sets of return times and host indices of ω, defined on the construction of the partition. Note that for $i = 1, \ldots, \gamma$, $f_a^{z_i}(\omega) \subset I^+_{m_i,k_i}$. Let $\sigma_i = f_a^{z_i}(\omega)$, $p_i = p(m_i)$, $x_i = f_a^i(x)$ and $y_i = f_a^i(y)$.

Observe that

$$\left|\frac{(f_a^n)'(x)}{(f_a^n)'(y)}\right| = \prod_{j=0}^{n-1}\left|\frac{f_a'(x_j)}{f_a'(y_j)}\right| = \prod_{j=0}^{n-1}\left|\frac{x_j}{y_j}\right| \leq \prod_{j=0}^{n-1}\left(1 + \left|\frac{x_j - y_j}{y_j}\right|\right)$$

Hence the result is proved if we manage to bound uniformly

$$S = \sum_{j=0}^{n-1}\left|\frac{x_j - y_j}{y_j}\right|.$$

For the moment assume that $n \leq z_\gamma + p_\gamma - 1$.

We first estimate the contribution of the free period between z_{q-1} and z_q for the sum S

$$F_q = \sum_{j=z_{q-1}+p_{k-1}}^{z_q-1}\left|\frac{x_j - y_j}{y_j}\right| \leq \sum_{j=z_{q-1}+p_{k-1}}^{z_q-1}\left|\frac{x_j - y_j}{\delta}\right|$$

For $j = z_{q-1} + p_{k-1}, \cdots, z_q - 1$ we have

$$\begin{aligned}
\lambda(\sigma_q) &\geq |f_a^{z_q-j}(x_j) - f_a^{z_q-j}(y_j)| \\
&= \left|(f_a^{z_q-j})'(\zeta)\right| \cdot |x_j - y_j|, \text{for some } \zeta \text{ between } x_j \text{ and } y_j \\
&\geq \mathrm{e}^{c_0(z_q-j)}|x_j - y_j|, \text{ by Lemma 2.1}
\end{aligned}$$

and so

$$\begin{aligned}
F_q &\leq \sum_{j=z_{q-1}+p_{k-1}}^{z_q-1} \mathrm{e}^{-c_0(z_q-j)} \cdot \frac{\lambda(\sigma_q)}{\delta} \\
&\leq \sum_{j=1}^{\infty} \mathrm{e}^{-cj} \cdot \frac{\lambda(I_{m_q})}{\delta} \cdot \frac{\lambda(\sigma_q)}{\lambda(I_{m_q})} \\
&\leq a_1 \cdot \frac{\lambda(\sigma_q)}{\lambda(I_{m_q})} \text{ for some constant } a_1 = a_1(c).
\end{aligned}$$

The contribution of the return z_q is

$$\left|\frac{x_{z_q} - y_{z_q}}{y_{z_q}}\right| \leq \frac{\lambda(\sigma_q)}{\mathrm{e}^{-|m_q|-2}} \leq a_2 \cdot \frac{\lambda(\sigma_q)}{\lambda(I_{m_q})} \text{ where } a_2 \text{ is a constant.}$$

Finally, let us compute the contribution of bound periods

$$B_q=\sum_{j=1}^{p_q-1}\left|\frac{x_{z_q+j}-y_{z_q+j}}{y_{z_q+j}}\right|$$

We have that

$$\begin{aligned}|x_{z_q+j}-y_{z_q+j}| &= |(f_a^j)'(\zeta)|\cdot|x_{z_q}-y_{z_q}|, \text{for some } \zeta \text{ between } x_{z_q} \text{ and } y_{z_q}\\ &= \left|(f_a^{j-1})'\left(f_a(\zeta)\right)\right|\cdot|f_a'(\zeta)|\cdot|x_{z_q}-y_{z_q}|\\ &= \left|(f_a^{j-1})'\left(f_a(\zeta)\right)\right|\cdot 2a|\zeta|\cdot|x_{z_q}-y_{z_q}|\\ &\le B_1|(f_a^{j-1})'(\xi_1)|\cdot 2a\mathrm{e}^{-|m_q|+1}\cdot\lambda(\sigma_q).\end{aligned}$$

On the other hand, we have

$$|y_{z_q+j}-\xi_j|=|(f_a^{j-1})'(\theta)|\cdot|y_{z_q+1}-\xi_1|$$

for some $\theta\in[y_{z_q+1},\xi_1]$. Noting that $[y_{z_q+1},\xi_1]\subset f_a\left(U^+_{|m_q|}\right)$, we apply Lemma 2.3 and get

$$\begin{aligned}|y_{z_q+j}-\xi_j| &\ge \frac{1}{B_1}|(f_a^{j-1})'(\xi_1)|\cdot|y_{z_q+1}-\xi_1|\\ &= \frac{1}{B_1}|(f_a^{j-1})'(\xi_1)|\cdot 2ay_{z_q}^2\\ &\ge \frac{1}{B_1}|(f_a^{j-1})'(\xi_1)|\cdot 2a\mathrm{e}^{-2|m_q|-4}.\end{aligned}$$

Combining what we know about $|x_{z_q+j}-y_{z_q+j}|$ and $|y_{z_q+j}-\xi_j|$ we obtain

$$\begin{aligned}\frac{|x_{z_q+j}-y_{z_q+j}|}{|y_{z_q+j}|} &= \frac{|x_{z_q+j}-y_{z_q+j}|}{|y_{z_q+j}-\xi_j|}\cdot\frac{|y_{z_q+j}-\xi_j|}{|y_{z_q+j}|}\\ &\le B_1^2\frac{\mathrm{e}^5}{\mathrm{e}^{-|m_q|}}\cdot\lambda(\sigma_q)\cdot\frac{|y_{z_q+j}-\xi_j|}{|y_{z_q+j}|}\\ &\le B_1^2\cdot\mathrm{e}^5\cdot\frac{\lambda(\sigma_q)}{\lambda(I_{m_q})}\cdot\frac{\mathrm{e}^{-\beta j}}{\mathrm{e}^{-\alpha j}-\mathrm{e}^{-\beta j}}\end{aligned}$$

since

$$|y_{z_q+j}|\ge|\xi_j|-|y_{z_q+j}-\xi_j|\ge\mathrm{e}^{-\alpha j}-\mathrm{e}^{-\beta j}.$$

Clearly,

$$\sum_{j=1}^{\infty}\frac{\mathrm{e}^{-\beta j}}{\mathrm{e}^{-\alpha j}-\mathrm{e}^{-\beta j}}<\infty$$

and, therefore,

$$B_q\le a_3\cdot\frac{\lambda(\sigma_q)}{\lambda(I_{m_q})}$$

for some constant $a_3=a_3(\alpha-\beta)$.

From the estimates obtained above, we get

$$S \leq a_4 \cdot \sum_{q=0}^{\gamma} \frac{\lambda(\sigma_q)}{\lambda(I_{m_q})}, \text{ where } a_4 = a_1 + a_2 + a_3.$$

Defining $q(m) = \max\{q : m_q = m\}$ and using the fact that $\lambda(\sigma_{q+1}) \geq 2\lambda(\sigma_q)$ (lemma 4.1 part 1), we can easily see that

$$\sum_{\{q:m_q=m\}} \lambda(\sigma_q) \leq 2\lambda(\sigma_{q(m)}),$$

and so

$$\sum_{q=0}^{\gamma} \frac{\lambda(\sigma_q)}{\lambda(I_{m_q})} \leq \sum_{m\geq\Delta} \frac{1}{\lambda(I_m)} \sum_{\{q:m_q=m\}} \lambda(\sigma_q) \leq \sum_{m\geq\Delta} \frac{2\lambda(\sigma_{q(m)})}{\lambda(I_m)}.$$

Since

$$\frac{\lambda(\sigma_{q(m)})}{\lambda(I_m)} \leq \frac{10}{m^2},$$

it follows that

$$\sum_{m\geq\Delta} \frac{2\lambda(\sigma_{q(m)})}{\lambda(I_m)} \leq 20 \sum_{m\geq\Delta} \frac{1}{m^2},$$

which proves that S is uniformly bounded .

Now, if $n \geq z_\gamma + p_\gamma$ we are left with a last piece of free period to study:

$$F_{\gamma+1} = \sum_{j=z_\gamma+p_\gamma}^{n} \left| \frac{x_j - y_j}{y_j} \right|$$

We consider two cases. In the first one we suppose that $|f_a^n(\omega)| \leq \mathrm{e}^{-2\Delta}$. Proceeding as before we have for $j = z_\gamma + p_\gamma, \ldots, n-1$,

$$\begin{aligned} \lambda(\sigma_n) &\geq |f_a^{n-j}(x_j) - f_a^{n-j}(y_j)| \\ &= \left|(f^{n-j})'(\zeta)\right| \cdot |x_j - y_j|, \text{for some } \zeta \text{ between } x_j \text{ and } y_j \\ &\geq \mathrm{e}^{-(\Delta+1)}\mathrm{e}^{c_0(n-j)}|x_j - y_j|, \text{ by Lemma 2.1 part 1.} \end{aligned}$$

So,

$$\begin{aligned} F_{\gamma+1} &\leq \sum_{j=z_\gamma+p_\gamma}^{n} \frac{\mathrm{e}^{\Delta+1}\mathrm{e}^{-c_0(n-j)}\lambda(\sigma_n)}{\delta} \\ &\leq \sum_{j=z_\gamma+p_\gamma}^{n} \mathrm{e}^{2\Delta+1}\mathrm{e}^{-c_0(n-j)}\mathrm{e}^{-2\Delta} \\ &\leq e \sum_{j=1}^{\infty} \mathrm{e}^{-cj} \leq a_5, \end{aligned}$$

where a_5 is constant.

In the second case we assume that $|f_a^n(\omega)| > \mathrm{e}^{-2\Delta}$. Let q_1 be the first integer such that $q_1 \geq z_\gamma + p_\gamma$, $|f_a^{q_1}(\omega)| > \mathrm{e}^{-2\Delta}$. From the previous argumentation we have that

$$\left|\frac{(f_a^{q_1})'(x)}{(f_a^{q_1})'(y)}\right| \leq C.$$

At this point we consider the time-interval $[q_1, q_2-1]$ (eventually empty) defined to be the largest interval such that $i \in [q_1, q_2-1] \Rightarrow y_i \notin U_1$. Then, using lemma 2.1 part 3 (here we use for the first time the hypothesis $f_a^n(\omega) \subset U_1$),

$$\sum_{i=q_1}^{q_2-1} \frac{|x_i - y_i|}{|y_i|} \leq e \sum_{i=q_1}^{q_2-1} |x_i - y_i| \leq 3 \sum_{i=q_1}^{q_2-1} \tfrac{5}{4} \mathrm{e}^{-c_0(n-1)} |f_a^n(\omega)|$$

$$\leq \tfrac{15}{2} \sum_{i=1}^{\infty} \mathrm{e}^{-ci} \leq a_6,$$

where a_6 is a constant.

If $q_2 = n$ the lemma is proved. Otherwise writing:

$$\left|\frac{(f_a^n)'(x)}{(f_a^n)'(y)}\right| = \left|\frac{(f_a^{n-q_2})'(x_{q_2})}{(f_a^{n-q_2})'(y_{q_2})}\right| \left|\frac{(f_a^{q_2})'(x)}{(f_a^{q_2})'(y)}\right|,$$

we observe that in order to obtain the result we need only to bound the first factor. We do this considering, again, two cases:

1. $x_{q_2} \geq \frac{1}{2}$. Then since $|y_{q_2}| \leq \mathrm{e}^{-1}$ (by definition of q_2), we have $|x_{q_2} - y_{q_2}| \geq \frac{1}{10}$. Therefore, by lemma 2.1 part 3

$$\tfrac{4}{5} \mathrm{e}^{c_0(n-q_2)} \tfrac{1}{10} \leq |f_a^n(\omega)| \leq 1,$$

which implies that $n - q_2 \leq \frac{3}{2} \log\left(\frac{25}{2}\right)$ (remember that by hypothesis $c_0 \geq \frac{2}{3}$).

Taking into account the facts: $|(f_a^{n-q_2})'(x_{q_2})| \leq 4^{n-q_2}$ and $|(f_a^{n-q_2})'(y_{q_2})| \geq \frac{4}{5}\mathrm{e}^{c_0(n-q_2)}$, we have

$$\left|\frac{(f_a^{n-q_2})'(x_{q_2})}{(f_a^{n-q_2})'(y_{q_2})}\right| \leq a_7,$$

for some constant a_7.

2. $x_{q_2} < \frac{1}{2}$. We can write (see Lemma 2.2 of [Al92] or Lemma 3.3 of [Mo92] for details)

$$\left|\frac{(f_a^{n-q_2})'(x_{q_2})}{(f_a^{n-q_2})'(y_{q_2})}\right| = L \left|\frac{(g_a^{n-q_2})'(h^{-1}(x_{q_2}))}{(f_a^{n-q_2})'(h^{-1}(y_{q_2}))}\right|,$$

where

$$L = \sqrt{\frac{1 - (f_a^{n-q_2}(x_{q_2}))^2}{1 - x_{q_2}^2}} \sqrt{\frac{1 - y_{q_2}^2}{1 - (f_a^{n-q_2}(y_{q_2}))^2}} \leq \sqrt{\frac{1}{1 - \frac{1}{4}}} \sqrt{\frac{1}{1 - \mathrm{e}^{-2}}} \leq \frac{3}{4},$$

$h : [-1, 1] \to [-1, 1]$ is the homeomorphism that conjugates $f_2(x)$ to the tent map $1 - 2|x|$ and $g_a = h^{-1} \circ f_a \circ h$.

For the second factor, we have (see Lemma 3.1 of [Mo92] for details)

$$\left|\frac{(g_a^{n-q_2})'(h^{-1}(x_{q_2}))}{(f_a^{n-q_2})'(h^{-1}(y_{q_2}))}\right| \leq \left(\frac{2+\frac{3\pi}{\delta^3}(2-a)}{2-\frac{3\pi}{\delta^3}(2-a)}\right)^{n-q_2}.$$

Note that $|f_a^{q_1}(\omega)| > \mathrm{e}^{-2\Delta}$ and $\frac{4}{5}\mathrm{e}^{c_0(n-q_1)}|f_a^{q_1}(\omega)| \leq |f_a^n(\omega)| \leq 1$, from which we conclude that $n-q_2 \leq n-q_1 \leq 4\Delta$. So if a is sufficiently close to 2 in order to have

$$\left(\frac{2+\frac{3\pi}{\delta^3}(2-a)}{2-\frac{3\pi}{\delta^3}(2-a)}\right)^{4\Delta} \leq 2, \tag{4.7}$$

then

$$\left|\frac{(f_a^{n-q_2})'(x_{q_2})}{(f_a^{n-q_2})'(y_{q_2})}\right| \leq \frac{8}{3}.$$

□

5. Return depths and time between consecutive returns

In this section we justify the preponderance of the depths of essential returns over the depths of bound and inessential returns, stated in basic idea (I). We also get an upper bound for the elapsed time between two consecutive essential returns.

As we have already mentioned, there are three types of returns: essential, bounded and inessential, which we denote by t, u and v respectively. Remember that up to time n, the essential return that occurs at time t_i has depth η_i, for $i=1,\ldots,s_n$; each t_i might be followed by bounded returns $u_{i,j}$, $j=1,\ldots,u$ and these can be followed by inessential returns $v_{i,j}$, $j=1,\ldots,v$.

The following lemma states that the depth of an inessential return is not greater than the depth of the essential return that precedes it.

Lemma 5.1. *Suppose that t_i is an essential return for $\omega \in \mathcal{P}_{t_i}$, with $I_{\eta_i,k_i} \subset f_a^{t_i}(\omega) \subset I^+_{\eta_i,k_i}$. Then the depth of each inessential return occurring on $v_{i,j}$, $j=1,\ldots,v$ is not greater than η_i.*

Proof. By lemma 4.1 part 1 we have

$$\lambda\left\{f_a^{v_{i,j}}(\omega)\right\} \geq 2^j\lambda\left\{f_a^{t_i}(\omega)\right\} \geq 2^j\lambda\left(I_{\eta_i,k_i}\right)$$

Thus,

$$\lambda\left\{f_a^{v_{i,j}}(\omega)\right\} \geq \lambda\left\{I_{\eta_i,k_i}\right\} = \frac{\mathrm{e}^{-\eta_i}\left(1-\mathrm{e}^{-1}\right)}{\eta_i^2}.$$

But, since $v_{i,j}$ is an inessential return time we must have $f_a^{v_{i,j}}(\omega) \subset I_{m,k}$ for some $m \geq \Delta$; then, out of necessity, $m \leq \eta_i$, because $f_a^{v_{i,j}}(\omega)$ is too large to fit on some $I_{m,k}$ with $m > \eta_i$. □

In the next lemma, we prove a similar result for bounded returns.

Lemma 5.2. *Let t be a return time (either essential or inessential) for $\omega \in \mathcal{P}_t$, with $f_a^t(\omega) \subset I_{\eta,k}^+$. Let $p = p(\eta)$ be the bound period length associated to this return. Then, for all $x \in \omega$, if the orbit of x returns to U_Δ between t and $t+p$, then the depth of this bound return will not be grater than η, if Δ is sufficiently large.*

Proof. Consider a point $x \in \omega$. We will show that if Δ is large enough then $|f_a^{t+j}(x)| \geq \mathrm{e}^{-\eta}$, $\forall j \in \{1, \ldots, p-1\}$.

$$\left|f_a^j(1)\right| - \left|f_a^{t+j}(x)\right| \leq \left|f_a^{t+j}(x) - f_a^j(1)\right| \leq \mathrm{e}^{-\beta j}$$

which implies that

$$\begin{aligned}
\left|f_a^{t+j}(x)\right| &\geq \left|f_a^j(1)\right| - \mathrm{e}^{-\beta j} \overset{\text{(BA)}}{\geq} \mathrm{e}^{-\alpha j} - \mathrm{e}^{-\beta j} \geq \mathrm{e}^{-\alpha j}\left(1 - \mathrm{e}^{(\alpha-\beta)j}\right) \\
&\geq \mathrm{e}^{-\alpha j}\left(1 - \mathrm{e}^{(\alpha-\beta)}\right), \text{ since } \alpha - \beta < 0 \\
&\geq \mathrm{e}^{-\alpha p}\left(1 - \mathrm{e}^{(\alpha-\beta)}\right), \text{ since } j < p \\
&\geq \mathrm{e}^{-3\alpha\eta}\left(1 - \mathrm{e}^{(\alpha-\beta)}\right), \text{ since } p \leq 3\eta \text{ by lemma 2.3} \\
&\geq \mathrm{e}^{-4\alpha\eta}, \text{ if we choose a large } \Delta \text{ so that } 1 - \mathrm{e}^{\alpha-\beta} \geq \mathrm{e}^{-\alpha\eta} \\
&\geq \mathrm{e}^{-\eta}, \text{ since } \alpha < \frac{1}{4}
\end{aligned}$$

□

The next lemma gives an upper bound for the time we have to wait between two essential return situations.

Lemma 5.3. *Suppose t_i is an essential return for $\omega \in \mathcal{P}_{t_i}$, with $I_{\eta_i,k_i} \subset f_a^{t_i}(\omega) \subset I_{\eta_i,k_i}^+$. Then the next essential return situation t_{i+1} satisfies:*

$$t_{i+1} - t_i < 5\left|\eta_i\right|.$$

Proof. Let $v_{i,1} < \ldots < v_{i,v}$ denote the inessential returns between t_i and t_{i+1}, with host intervals $I_{\eta_{i,1},k_{i,1}}, \ldots, I_{\eta_{i,v},k_{i,v}}$, respectively. We also consider $v_{i,0} = t_i$; $v_{i,v+1} = t_{i+1}$; for $j = 0, \ldots, v+1$, $\sigma_j = f_a^{v_{i,j}}(\omega)$ and for $j = 0, \ldots, v$, $q_j = v_{i,j+1} - (v_{i,j} + p_j)$, where p_j is the length of the bound period associated to the return $v_{i,j}$.

We consider two different cases: $v = 0$ and $v > 0$.

(1) $v = 0$

In this situation $t_{i+1} - t_i = p_0 + q_0$. Applying lemma 4.1 part 2b we get that

$$|\sigma_1| \geq \mathrm{e}^{-5\beta|\eta_i|}\mathrm{e}^{c_0 q_0 - (\Delta+1)}.$$

Taking into account the fact that $|\sigma_1| \leq 2$, we have

$$
\begin{aligned}
&c_0 q_0 \leq 1 + 5\beta|\eta_i| + \Delta + 1 \\
&q_0 \leq 8\beta|\eta_i| + \frac{3}{2}\Delta + 3, \text{ since } c_0 \geq \frac{2}{3} \\
&q_0 \leq 9\beta|\eta_i| + \frac{3}{2}\Delta, \text{for } \Delta \text{ large enough so that } \beta|\eta_i| > 3.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
t_{i+1} - t_i &= p_0 + q_0 \\
&\leq 3|\eta_i| + 9\beta|\eta_i| + \frac{3}{2}\Delta \\
&\leq 4|\eta_i| + \Delta, \text{since } 9\beta < \frac{1}{2} \\
&\leq 5|\eta_i|.
\end{aligned}
$$

(2) $v > 0$

In this case, $t_{i+1} - t_i = \sum_{j=0}^{v}(p_j + q_j)$. We separate this sum into three parts and control each separately:

$$
t_{i+1} - t_i = p_0 + \left(\sum_{j=1}^{v-1} p_j + \sum_{j=0}^{v-1} q_j\right) + (p_v + q_v)
$$

(i) For p_0 we have by lemma 2.3 that $p_0 \leq 3|\eta_i|$.

(ii) By lemma 4.1 we get

$$
|\sigma_1| \geq \mathrm{e}^{c_0 q_0}\mathrm{e}^{-5\beta|\eta_i|} \text{ and } \frac{|\sigma_{j+1}|}{|\sigma_j|} \geq \mathrm{e}^{c_0 q_j}\mathrm{e}^{(1-5\beta)|\eta_{i,j}|},
$$

for $j = 1, \ldots, v-1$. Now, we observe that $p_j \leq 3|\eta_{i,j}| \leq 4(1-5\beta)|\eta_{i,j}|$ and $q_j \leq 4c_0 q_j$, for all $j = 0, \ldots, v$. This means that controlling the second parcel resumes to bound

$$
\sum_{j=1}^{v-1}(1-5\beta)|\eta_{i,j}| + \sum_{j=0}^{v-1} c_0 q_j. \tag{5.1}
$$

We achieve our goal by noting that (5.1) corresponds to the growth rate of the size of the σ_j, which cannot be very large, since every σ_j, $j = 1, \ldots, v$ is contained in some $I_{m,k} \subset U_\Delta$. Writing

$$
|\sigma_v| = |\sigma_1| \prod_{j=1}^{v-1} \frac{|\sigma_{j+1}|}{|\sigma_j|},
$$

and taking into account that $\sigma_v \in I_{\eta_{i,v},k_{i,v}}$, with $|\eta_{i,v}| \geq \Delta$ and thus $|\sigma_v| \leq \mathrm{e}^{-(\Delta+1)}$, it follows that

$$
\exp\left\{-5\beta|\eta_i| + \sum_{j=0}^{v-1} c_0 q_j + \sum_{j=1}^{v-1}(1-5\beta)|\eta_{i,j}|\right\} \leq \exp\{-(\Delta+1)\}
$$

and consequently
$$\sum_{j=1}^{v-1}(1-5\beta)|\eta_{i,j}| + \sum_{j=0}^{v-1} c_0 q_j \leq 5\beta|\eta_i| - (\Delta+1)$$

(iii) For the last term $p_v + q_v$ we proceed in a very similar manner to what we did in the case $v = 0$. By lemma 4.1we have
$$\frac{|\sigma_{v+1}|}{|\sigma_v|} \geq e^{c_0 q_v - (\Delta+1)} e^{(1-4\beta)|\eta_{i,v}|} \geq e^{c_0 q_v - (\Delta+1)} e^{(1-5\beta)|\eta_{i,v}|}.$$
From part 1 of lemma 4.1 we have $|\sigma_v| \geq 2^{v-1}|\sigma_1| \geq |\sigma_1|$, from which we get
$$2 \geq |\sigma_{v+1}| \geq |\sigma_1| \frac{|\sigma_{v+1}|}{|\sigma_v|}$$
and consequently
$$\exp\{-5\beta|\eta_i| + c_0 q_v - (\Delta+1) + (1-5\beta)|\eta_{i,v}|\} \leq e^{\log 2}$$
implying
$$c_0 q_v + (1-5\beta)|\eta_{i,v}| \leq \Delta + 2 + 5\beta|\eta_i|.$$
Putting together the three parts we get
$$\begin{aligned}
t_{i+1} - t_i &= p_0 + \left(\sum_{j=1}^{v-1} p_j + \sum_{j=0}^{v-1} q_j\right) + (p_v + q_v) \\
&\leq p_0 + 4\left\{\sum_{j=1}^{v-1}(1-5\beta)|\eta_{i,j}| + \sum_{j=0}^{v-1} c_0 q_j + c_0 q_v + (1-5\beta)|\eta_{i,v}|\right\} \\
&\leq 3|\eta_i| + 4\{5\beta|\eta_i| - (\Delta+1) + (\Delta+1) + 1 + 5\beta|\eta_i|\} \\
&\leq 3|\eta_i| + 40\beta|\eta_i| + 4 \\
&\leq 4|\eta_i|.
\end{aligned}$$

□

The next two propositions allow us to obtain a bound for $T_n(x)$ (see (2.1) for definition) by a quantity proportional to $\frac{1}{n}F_n(x)$ (defined in (3.8)).

In the proof of the following proposition we will use directly and for the first time the condition known as the *free assumption* for the critical orbit. This condition essentially asserts that the set of Benedicks-Carleson parameters is built in such a way that the amount of time spent by the critical orbit in bound periods totally makes up a small fraction of the whole time (see [BC91, Section 2] or [Mo92, condition FA(n)]).

Proposition 5.4. *Let t be a free return time (either essential or inessential) for $\omega \in \mathcal{P}_t$ with $f_a^t(\omega) \subset I_{\eta,k}^+$. Let $p = p(\eta)$ be the bound period associated with this return. Let S denote the sum of the depths of all the bound returns plus the depth of the return that originated the bound period. Then $S \leq C_3\eta$, with constant $C_3 = C_3(\alpha)$.*

Proof. Recall that by lemma 2.3 we know that $\frac{2}{3}\eta \leq p \leq 3\eta$. Let $x \in \omega$. We say that a bound return is of *level* i if, at the moment of this bound return, x has already initiated exactly i bindings to the critical point ξ_0 and all of them are still active. By active we mean that the respective bound periods have not finished yet. To illustrate, suppose that u_1 is the first time between t and $t+p$ that the orbit of x enters U_Δ. Obviously, at this moment, the only active binding to ξ_0 is the one initiated at time t. Thus, u_1 is a bound return of level 1. Now, at time u_1, the orbit of x establishes a new binding to the critical point which ends at the end of the corresponding bound period that we denote by p_1 which depends on the depth η_1 of the bound return in question. During the period from u_1 to u_1+p_1 new returns may happen and their level is at least 2 since there are at least 2 active bindings: the one initiated at t and the one initiated at u_1. If $u_1+p_1 < t+p$ then new bound returns of level 1 may occur after u_1+p_1.

We may redefine the notion of bound period so that the bound periods are nested (see [BC91], section 6.2). This means that we may suppose that no binding of level i extends beyond the bound period of level $i-1$ during which it was initiated.

Taking into account the *free assumption* condition for the critical orbit we may assume that in a period of length $n \in \mathbb{N}$, the time spent by the critical orbit in bound periods is at most αn (see [Mo92, condition FA(n)]).

Since, when a point initiates a binding with ξ_0, it shadows the early iterates of the critical point, the same applies to any of these points $x \in \omega$ bounded to ξ_0. Thus in the period of time from t to $t+p$, the orbit of x can spend at most the fraction of time αp in bound periods. So if l denotes the number of bound returns of level 1, $u_1, \ldots, u_l$ their instants of occurrence, $\eta_1, \ldots \eta_l$ their respective depths and $p_1, \ldots, p_l$ their respective bound periods, then we have by lemma 2.3 and the above observation that:

$$\frac{2}{3}\sum_{i=1}^{l}\eta_i \leq \sum_{i=1}^{l}p_i \leq \alpha p \leq 3\alpha\eta$$

from where we easily obtain $\sum_{i=1}^{l}\eta_i \leq 5\alpha\eta$. The same argument applies to the bound returns of level 2 within the i-th bound period of level 1. So if l_i denotes the number of bound returns of level 2 within the i-th bound period of level 1, $u_{i1}, \ldots, u_{il_i}$ their instants of occurrence, $\eta_{i1}, \ldots \eta_{il}$ their respective depths and $p_{i1}, \ldots, p_{il}$ their respective bound periods, then we have

$$\frac{2}{3}\sum_{j=1}^{l_i}\eta_{ij} \leq \sum_{i=1}^{l_i}p_{ij} \leq \alpha p_i \leq 3\alpha\eta_i$$

from where we easily obtain $\sum_{i=1}^{l}\sum_{j=1}^{l_i}\eta_{ij} \leq (5\alpha)^2\eta$.

Thus a simple induction argument gives that

$$S \leq \sum_{i=0}^{\infty}(5\alpha)^i\eta \leq C_3\eta,$$

where

$$C_3 = \frac{1}{1-5\alpha}, \tag{5.2}$$

remember that by choice $5\alpha < 1$. □

Proposition 5.5. *Let t be an essential return time for $\omega \in \mathcal{P}_t$ with $I_{\eta,k} \subset f_a^t(\omega) \subset I_{\eta,k}^+$. Let p_0 denote the associated bound period. Let S denote the sum of the depths of all the free inessential returns before the next essential return situation. Then $S \le C_4\eta$, with constant $C_4 = C_4(\beta)$.*

Proof. Suppose that v is the number of inessential returns before the next essential return situation of ω, which occur at times $v_1, \dots, v_v$, with respective depths $\eta_1, \dots, \eta_v$ and respective bound periods $p_1, \dots, p_v$. Also denote by v_{v+1} the next essential return situation of ω. Let $\sigma_i = f_a^{v_i}(\omega)$.

By lemma 4.1 we get

$$|\sigma_1| \ge \mathrm{e}^{c_0 q_0} \mathrm{e}^{-5\beta|\eta|} \text{ and } \frac{|\sigma_{j+1}|}{|\sigma_j|} \ge \mathrm{e}^{c_0 q_i} \mathrm{e}^{(1-5\beta)|\eta_i|},$$

where $q_i = v_{i+1} - (v_i + p_i)$, for $i = 0, \dots, v$. We also know that $|\sigma_{v+1}| \le 2$.

Since

$$|\sigma_{v+1}| = |\sigma_1| \prod_{i=1}^{v} \frac{|\sigma_{i+1}|}{|\sigma_i|},$$

we have that

$$\exp\left\{ c_0 q_0 - 5\beta\eta + \sum_{i=1}^{v} (c_0 q_i + (1-5\beta)\eta_i) \right\} \le \mathrm{e},$$

from where we obtain that

$$\sum_{i=1}^{v} (c_0 q_i + (1-5\beta)\eta_i) \le 5\beta\eta + 1,$$

which easily implies that $S \le C_4\eta$, where

$$C_4 = \frac{6\beta}{1-5\beta} \tag{5.3}$$

. □

From these propositions we easily conclude that

$$T_n(x) \le \frac{C_5}{n} F_n(x)$$

with

$$C_5 = C_5(\alpha, \beta) = (C_3 + C_3 C_4). \tag{5.4}$$

6. Probability of an essential return reaching a certain depth

Now, that we know that only the essential returns matter, we prove that the chances of very deep essential returns occurring are very small. In fact, the probability of an essential return hitting the depth of ρ will be shown to be less than $\mathrm{e}^{-\tau\rho}$, with $\tau > 0$.

We must make our statements more precise and we begin by defining a probability space. We define the probability measure λ^* on I by renormalizing the Lebesgue measure so that $\lambda^*(I) = 1$. We may now speak of expectations $E(\cdot)$, events and their probability of occurrence.

For each $x \in I$, let $u_n(x)$ denote the number of essential return situations of x between 1 and n, $s_n(x)$ be the number those which are actual essential return times and sd_n the number of the latter that correspond to deep essential returns of the orbit of x with return depths above a threshold $\Theta \geq \Delta$. Observe that $u_n(x) - s_n(x)$ is the exact number of escaping situations of the orbit of x, up to n.

Given the integers $0 \leq s \leq \frac{3n}{2\Theta}$, $s \leq u \leq n$ and s integers $\rho_1, \ldots, \rho_s$, each greater than or equal to Θ, we define the event:

$$A^{u,s}_{\rho_1,\ldots,\rho_s}(n) = \left\{ x \in I \ : \ u_n(x) = u,\, sd_n(x) = s, \begin{array}{r} \text{and the depth of the i-th deep essen-} \\ \text{tial return is } \rho_i \ \forall i \in \{1, \ldots, s\} \end{array} \right\}.$$

Remark 6.1. Observe that the upper bound $\frac{3n}{2\Theta}$ for the number of deep essential returns up to time n derives from the fact that each deep essential return originates a bound period of length at least $\frac{2}{3}\Theta$ (see lemma 2.3). Since during the bound periods there cannot be any essential return, the number of deep essential returns occurring in a period of length n is at most $\frac{n}{\frac{2}{3}\Theta}$.

Proposition 6.2. *Given the integers $0 \leq s \leq \frac{3n}{2\Theta}$ and $s \leq u \leq n$, consider s integers $\rho_1, \ldots, \rho_s$, each greater than or equal to Θ. If Θ is large enough, then*

$$\lambda^* \left(A^{u,s}_{\rho_1,\ldots,\rho_s}(n) \right) \leq \binom{u}{s} \mathit{Exp} \left\{ -(1 - 6\beta) \sum_{i=1}^{s} \rho_i \right\}$$

Proof. Fix $n \in \mathbb{N}$ and take $\omega_0 \in \mathcal{P}_0$. Note that the functions u_n, s_n and sd_n are constant in each $\omega \in \mathcal{P}_n$. Let $\omega \in \omega_0 \cap \mathcal{P}_n$ be such that $u_n(\omega) = u$. Then, there is a sequence $1 \leq t_1 \leq \ldots \leq t_u \leq n$ of essential return situations. Let ω_i denote the element of the partition $\mathcal{P}_{t_i}$ that contains ω. We have $\omega_0 \supset \omega_1 \supset \ldots \supset \omega_u = \omega$. Consider that $\omega_j = \emptyset$ whenever $j > u$. For each $j \in \{0, \ldots, n\}$ we define the set:

$$Q_j = \bigcup_{\omega \in \mathcal{P}_n \cap \omega_0} \omega_j,$$

and its partition

$$\mathcal{Q}_j = \{\omega_j : \omega \in \mathcal{P}_n \cap \omega_0\}.$$

Let $\omega \in \mathcal{P}_n$ be such that $sd_n(\omega) = s$. Then, we may consider $1 \leq r_1 \leq \ldots \leq r_s \leq u$ with r_i indicating that the i-th deep essential return occurs in the r_i-th essential return situation.

Now, set $V(0) = Q_0 = \omega_0$. Fix s integers $1 \le r_1 \le \ldots \le r_s \le u$. Next, for each $j \le u$ we define recursively the sets $V(j)$. Although the set $V(u)$ will depend on the fixed integers $1 \le r_1 \le \ldots \le r_s \le u$, we do not indicate this so that the notation is not overloaded. Suppose that $V(j-1)$ is already defined and $r_{i-1} < j < r_i$. Then, we set

$$V(j) = \bigcup_{\omega \in \mathcal{Q}_j} \omega \bigcap f_a^{-t_j}(I - U_\Theta) \bigcap V(j-1).$$

If $j = r_i$ then we define

$$V(j) = \bigcup_{\omega \in \mathcal{Q}_j} \omega \bigcap f_a^{-t_j}(I_{\rho_i} \cup I_{-\rho_i}) \bigcap V(j-1)$$

Observe that for every $j \in \{1, \ldots, u\}$ we have $\frac{|V(j)|}{|V(j-1)|} \le 1$. Therefore, we concentrate in finding a better estimate for $\frac{|V(r_i)|}{|V(r_i-1)|}$. Consider that $\omega_{r_i} \in \mathcal{Q}_{r_i} \cap V(r_i - 1)$ and let $\omega_{r_i-1} \in \mathcal{Q}_{r_i-1} \cap V(r_i - 1)$ contain ω_{r_i}. We have to consider two situations depending on whether t_{r_i-1} is an escaping situation or an essential return.

Let us suppose first that t_{r_i-1} was an essential return with return depth η. Then,

$$\begin{aligned}
\frac{|\omega_{r_i}|}{|\omega_{r_i-1}|} &\le \frac{|\omega_{r_i}|}{|\widehat{\omega}_{r_i-1}|}, \quad \text{where } \widehat{\omega}_{r_i-1} = \omega_{r_i-1} \cap f_a^{-t_{r_i}}(U_1) \\
&\le C \frac{\left| f_a^{t_{r_i}}(\omega_{r_i}) \right|}{\left| f_a^{t_{r_i}}(\widehat{\omega}_{r_i-1}) \right|}, \text{ by the mean value theorem and lemma 4.3} \\
&\le C \frac{2\mathrm{e}^{-\rho_i}}{\mathrm{e}^{-5\beta\eta}}, \text{ by lemma 4.1 part 3b and definition of } \omega_{r_i}
\end{aligned}$$

Note that when $r_{i-1} = r_i - 1$ then $\eta = \rho_{i-1} \ge \Theta$. If, on the other hand, $r_{i-1} > r_i - 1$ then t_{r_i-1} is an essential return with depth $\eta < \Theta \le \rho_{i-1}$. Then in both situations we have

$$\frac{|\omega_{r_i}|}{|\omega_{r_i-1}|} \le 2C \frac{\mathrm{e}^{-\rho_i}}{\mathrm{e}^{-5\beta\rho_{i-1}}}.$$

When t_{r_i-1} is an escape situation instead of using lemma 4.1 we can use lemma 4.2 and obtain

$$\frac{|\omega_{r_i}|}{|\omega_{r_i-1}|} \le 2C \frac{\mathrm{e}^{-\rho_i}}{\mathrm{e}^{-\beta\Delta}} \le 2C \frac{\mathrm{e}^{-\rho_i}}{\mathrm{e}^{-5\beta\rho_{i-1}}}.$$

Observe also that if $\widehat{\omega}_{r_i-1} \ne \omega_{r_i-1}$ then, because we are assuming that $\omega_{r_i} \ne \emptyset$, we have $\lambda\left(f_a^{t_{r_i}}(\widehat{\omega}_{r_i-1})\right) \ge \mathrm{e}^{-1} - \mathrm{e}^{-\Theta} \ge \mathrm{e}^{-5\beta\rho_{i-1}}$, for large Θ.

At this point we have

$$\begin{aligned}|V(r_i)| &= \sum_{\omega_{r_i}\in\mathcal{Q}_{r_i}\cap V(r_i-1)} \frac{|\omega_{r_i}|}{|\omega_{r_i-1}|}|\omega_{r_i-1}| \\ &\leq 2C\mathrm{e}^{-\rho_i}\mathrm{e}^{5\beta\rho_{i-1}} \sum_{\omega_{r_i}\in\mathcal{Q}_{r_i}\cap V(r_i-1)} |\omega_{r_i-1}| \\ &\leq 2C\mathrm{e}^{-\rho_i}\mathrm{e}^{5\beta\rho_{i-1}}|V(r_i-1)|.\end{aligned}$$

We are now in conditions to obtain that

$$|V(u)| \leq (2C)^s \mathrm{Exp}\left\{-(1-5\beta)\sum_{i=1}^{s}\rho_i\right\}\mathrm{e}^{5\beta\rho_0}|V(0)|$$

where ρ_0 is given by the interval $\omega_0 \in \mathcal{P}_0$. If $\omega_0 = I_{(\eta_0,k_0)}$ with $|\eta_0| \geq \Delta$ and $1 \leq k_0 \leq \eta_0^2$, then $\rho_0 = |\eta_0|$. If $\omega_0 = (\delta, 1]$ or $\omega_0 = [-1, -\delta)$, then we can take $\rho_0 = 0$.

Now, we have to take into account the number of possibilities of having the occurrence of the event $V(u)$ implying the occurrence of the event $A^{u,s}_{\rho_1,\ldots,\rho_s}(n)$. The number of possible configurations related with the different values that the integers $r_1, \ldots r_s$ can take is $\binom{u}{s}$. Hence, it follows that

$$\begin{aligned}\lambda^*\left(A^{u,s}_{\rho_1,\ldots,\rho_s}(n)\right) &\leq (2C)^s\binom{u}{s}\mathrm{Exp}\left\{-(1-5\beta)\sum_{i=1}^{s}\rho_i\right\}\sum_{\omega_o\in\mathcal{P}_0}\mathrm{e}^{5\beta|\rho_0|}|\omega_0| \\ &\leq (2C)^s\binom{u}{s}\mathrm{Exp}\left\{-(1-5\beta)\sum_{i=1}^{s}\rho_i\right\}\left(2(1-\delta)+\sum_{|\eta_0|\geq\Delta}\mathrm{e}^{5\beta\eta_0}\mathrm{e}^{-|\eta_0|}\right) \\ &\leq 3(2C)^s\binom{u}{s}\mathrm{Exp}\left\{-(1-5\beta)\sum_{i=1}^{s}\rho_i\right\}, \quad \text{for } \Delta \text{ large enough} \\ &\leq \binom{u}{s}\mathrm{Exp}\left\{-(1-6\beta)\sum_{i=1}^{s}\rho_i\right\}.\end{aligned}$$

The last inequality results from the fact that $s\Theta \leq \sum_{i=1}^{s}\rho_i$ and the freedom to choose a sufficiently large Θ. □

Fix $n \in \mathbb{N}$, the integers $1 \leq s \leq \frac{n}{\frac{2}{3}\Theta}$, $s \leq u \leq n$ and integer $j \leq s$. Given an integer $\rho \geq \Theta$, consider the event

$$A^{u,s}_{\rho,j}(n) = \left\{ x \in I \;:\; u_n(x) = u,\, sd_n(x) = s, \;\; \begin{array}{r}\text{and the depth of the j-th deep}\\ \text{essential return is } \rho\end{array} \right\}.$$

Corollary 6.3. *If Δ is large enough, then*

$$\lambda^*\left(A^{u,s}_{\rho,j}(n)\right) \leq \binom{u}{s} e^{-(1-6\beta)\rho}$$

Proof. Since $A^{u,s}_{\rho,j}(n) = \bigcup_{\substack{\rho_i\geq\Theta\\ i\neq j}} A^{u,s}_{\rho_1,\ldots,\rho_{j-1},\rho,\rho_{j+1},\ldots,\rho_s}(n)$, then by proposition 6.2 we have

$$\lambda^*\left(A^{u,s}_{\rho,j}(n)\right) \leq \binom{u}{s}\mathrm{e}^{-(1-6\beta)\rho}\left(\sum_{\eta=\Theta}^{\infty}\mathrm{e}^{-(1-6\beta)\eta}\right)^{s-1} \leq \binom{u}{s}\mathrm{e}^{-(1-6\beta)\rho},$$

as long as Θ is sufficiently large so that $\sum_{\eta=\Theta}^{\infty}\mathrm{e}^{-(1-6\beta)\eta} \leq 1$. □

Remark 6.4. Observe that the bound for the probability of the event $A^{u,s}_{\rho,j}(n)$ does not depend on the $j \leq s$ chosen.

Remark 6.5. Observe that proposition 6.2 and corollary 6.3 also apply when $\Theta = \Delta$ in which case we have $sd_n = s_n$.

7. Non-uniform expansion

According to section 3 to finish the proof we only need to show that

$$\lambda\left(E_1(n)\right) \leq \mathrm{e}^{-\tau_1 n}, \quad \forall n \geq N_1^*$$

for some constant $\tau_1(\alpha,\beta) > 0$ and an integer $N_1^* = N_1^*(\Delta,\tau_1)$.

For each $x \in I$, recall that $u_n(x)$ denotes the number of essential return situations of x between 1 and n, and $s_n(x)$ the number of those which correspond to essential returns of the orbit of x. In this section we consider that the threshold $\Theta = \Delta$. Also remember that $u_n(x) - s_n(x)$ is the exact number of escaping situations the orbit of x goes through until the time n.

We define the following events:

$$A^{u,s}_{\rho}(n) = \left\{ x \in I:\ u_n(x) = u,\ s_n(x) = s \text{ and there is one essential return} \atop \text{reaching the depth } \rho \right\},$$

for fixed $n \in \mathbb{N}$, $s \leq n$ and $\rho \geq \Delta$;

$$A_\rho(n) = \left\{x \in I : \exists t \leq n : t \text{ is essential return time and } |f_a^t(x)| \in I_\rho\right\},$$

for fixed n and $\rho \geq \Delta$.

Now, because $A^{u,s}_{\rho}(n) = \bigcup_{j=1}^{s} A^{u,s}_{\rho,j}(n)$, by corollary 6.3, we have

$$\lambda^*\left(A^{u,s}_{\rho}(n)\right) \leq \sum_{j=1}^{s}\lambda^*\left(A^{u,s}_{\rho,j}(n)\right) \leq s\binom{u}{s}\mathrm{e}^{-(1-6\beta)\rho}. \tag{7.1}$$

Observing that $A_\rho(n)=\bigcup_{s=1}^{\frac{3n}{2\Delta}}\bigcup_{u=s}^{n}A_\rho^s(n)$, then by (7.1) we get

$$\begin{aligned}\lambda^*\left(A_\rho(n)\right) &\le \sum_{s=1}^{\frac{3n}{2\Delta}}\sum_{u=s}^{n}\lambda^*\left(A_\rho^{u,s}(n)\right)\le \sum_{s=1}^{\frac{3n}{2\Delta}}\sum_{u=s}^{n} s\binom{u}{s}\mathrm{e}^{-(1-6\beta)\rho}\\ &\le \mathrm{e}^{-(1-6\beta)\rho}\sum_{s=1}^{\frac{3n}{2\Delta}} s\sum_{u=s}^{n}\binom{n}{s}\le n\mathrm{e}^{-(1-6\beta)\rho}\sum_{s=1}^{\frac{3n}{2\Delta}} s\binom{n}{s}\\ &\le n\binom{n}{\frac{3n}{2\Delta}}\mathrm{e}^{-(1-6\beta)\rho}\sum_{s=1}^{\frac{3n}{2\Delta}} s\le \frac{4n^3}{\Delta}\binom{n}{\frac{3n}{2\Delta}}\mathrm{e}^{-(1-6\beta)\rho}.\end{aligned}$$

By the Stirling formula, we have

$$\sqrt{2\pi m}m^m\mathrm{e}^{-m}\le m!\le\sqrt{2\pi m}m^m\mathrm{e}^{-m}\left(1+\frac{1}{4m}\right),$$

which implies that

$$\binom{n}{\frac{3n}{2\Delta}}\le \mathrm{const}\frac{(n)^n}{(n-\frac{3n}{2\Delta})^{n-\frac{3n}{2\Delta}}(\frac{3n}{2\Delta})^{\frac{3n}{2\Delta}}}.$$

So, if we choose Δ large enough we have

$$\binom{n}{\frac{3n}{2\Delta}}\le \mathrm{const}\left(\left(1+\frac{\frac{3}{2\Delta}}{1-\frac{3}{2\Delta}}\right)\left(1+\frac{1-\frac{3}{2\Delta}}{\frac{3}{2\Delta}}\right)^{\frac{\frac{3}{2\Delta}}{1-\frac{3}{2\Delta}}}\right)^{(n-\frac{3n}{2\Delta})}\le \mathrm{const}\mathrm{e}^{h(\Delta)n},$$

where $h(\Delta)\to 0$, as $\Delta\to\infty$. The last inequality derives from the fact that each factor in the middle expression can be made arbitrarily close to 1 by taking Δ sufficiently large.

Since we know, by lemmas 5.1 and 5.2, that the depths of inessential and bound returns are not greater than the depth of the essential return preceding them we have, for all $n\ge N_1'$, where N_1' is such that $\alpha N_1'\ge\Delta$,

$$E_1(n)=\left\{x\in I:\ \exists i\in\{1,\dots,n\},\ |f_a^i(x)|<\mathrm{e}^{-\alpha n}\right\}\subset\bigcup_{\rho=\alpha n}^{\infty}A_\rho(n).$$

Consequently, taking $\tau_1=\frac{(1-6\beta)\alpha}{4}$ and Δ large enough such that $h(\Delta)\le\frac{(1-6\beta)\alpha}{2}$

$$\begin{aligned}\lambda^*(E_1(n)) &\le \mathrm{const}\tfrac{4n^3}{\Delta}\mathrm{e}^{h(\Delta)n}\sum_{\rho=\alpha n}^{\infty}\mathrm{e}^{-(1-6\beta)\rho}\\ &\le \mathrm{const}'\tfrac{4n^3}{\Delta}\mathrm{e}^{h(\Delta)n}\mathrm{e}^{-(1-6\beta)\alpha n}\\ &\le \mathrm{const}'\tfrac{4n^3}{\Delta}\mathrm{e}^{-2\tau_1 n}\\ &\le \mathrm{e}^{-\tau_1 n},\end{aligned}$$

when $n\ge N_1^*$, where N_1^* is such that $N_1^*\ge N_1'$ and for all $n\ge N_1^*$ we have

$$\mathrm{const}'\ \tfrac{4n^3}{\Delta}\mathrm{e}^{-\tau_1 n}\le 1. \tag{7.2}$$

8. Slow recurrence to the critical set

As referred in section 3, we are left with the burden of having to show that for all $n \in \mathbb{N}$, and for a given ϵ, we may choose a small $\gamma = \mathrm{e}^{-\Theta}$ such that

$$\lambda^*\{E_2(n)\} \le \lambda^* \left\{ x : F_n(x) > \frac{\epsilon n}{C_5} \right\} \le \mathrm{e}^{-\tau_2 n},$$

in order to complete the proof.

We achieve this goal, by means of a large deviation argument. Essentially we show that the moment generating function of F_n is bounded above by $\mathrm{e}^{h(\Theta)n}$, where $h(\Theta) \xrightarrow{\Theta\to\infty} 0$; then we use the Tchebychev inequality to obtain the desired result.

Lemma 8.1. *Take $0 < t \le \frac{1-6\beta}{3}$. If Θ is sufficiently large, then there exists $N_2 \in \mathbb{N}$ such that for all $n \ge N_2$ we have $E\left(e^{tF_n}\right) \le e^{h(\Theta)n}$. Moreover $h(\Theta) \to 0$, as $\Theta \to \infty$.*

Proof.

$$\begin{aligned} E\left(\mathrm{e}^{tF_n}\right) = E\left(\mathrm{e}^{t\sum_{i=1}^{s}\eta_i}\right) &= \sum_{u,s,(\rho_1,\dots,\rho_s)} \mathrm{e}^{t\sum_{i=1}^{s}\rho_i} \lambda^*\left(A^{u,s}_{\rho_1,\dots,\rho_s}(n)\right) \\ &\le \sum_{u,s,(\rho_1,\dots,\rho_s)} \mathrm{e}^{t\sum_{i=1}^{s}\rho_i} \binom{u}{s} \mathrm{e}^{-3t\sum_{i=1}^{s}\rho_i}, \text{ by proposition 6.2} \\ &\le \sum_{u,s,R} \binom{u}{s} \zeta(s,R)\mathrm{e}^{-2tR}, \end{aligned}$$

where $\zeta(s,R)$ is the number of integer solutions of the equation $x_1+\ldots+x_s = R$ satisfying $x_i \ge \Theta$ for all i. We have

$$\zeta(s,R) \le \#\{\text{solutions of } x_1+\ldots+x_s = R, x_i \in \mathbb{N}_0\} = \binom{R+s-1}{s-1}.$$

By the Stirling formula, we may write

$$\sqrt{2\pi m}m^m\mathrm{e}^{-m} \le m! \le \sqrt{2\pi m}m^m\mathrm{e}^{-m}\left(1+\frac{1}{4m}\right),$$

which implies that

$$\binom{R+s-1}{s-1} \le \text{const}\frac{(R+s-1)^{R+s-1}}{R^R(s-1)^{s-1}}.$$

So, if we choose Θ large enough we have

$$\zeta(s,R) \le \left(\text{const}^{\frac{1}{R}}\left(1+\tfrac{s-1}{R}\right)\left(1+\tfrac{R}{s-1}\right)^{\frac{s-1}{R}}\right)^R \le \mathrm{e}^{tR}.$$

The last inequality derives from the fact that $s\Theta \le R$, and so each factor in the middle expression can be made arbitrarily close to 1 by taking Θ sufficiently large.

Continuing from where we stopped,

$$
\begin{aligned}
E\left(\mathrm{e}^{tF_n}\right) &\le \sum_{u,s,R} \binom{u}{s} \mathrm{e}^{tR}\mathrm{e}^{-2tR} \\
&\le \sum_{u,s,R} \binom{u}{s} \mathrm{e}^{-tR} \\
&\le \sum_{u,s} \binom{u}{s}, \quad \text{for } \Theta \text{ sufficiently large.}
\end{aligned}
$$

Now, we have

$$
\sum_{u,s} \binom{u}{s} \le \sum_{s=1}^{\frac{3n}{2\Theta}} \sum_{u=s}^{n} \binom{u}{s} \le n \sum_{s=1}^{\frac{3n}{2\Theta}} \binom{n}{s} \le n \sum_{s=1}^{\frac{3n}{2\Theta}} \binom{n}{\frac{3n}{2\Theta}} \le \frac{3n^2}{2\Theta} \binom{n}{\frac{3n}{2\Theta}}.
$$

Using the Stirling formula again, and arguing like in section 7 it follows that we may take $N_2 = N_2(\Theta) \in \mathbb{N}$ sufficiently large so that for all $n \ge N_2$ we obtain

$$
E\left(\mathrm{e}^{tF_n}\right) \le \mathrm{e}^{h(\Theta)n},
$$

where $h(\Theta) \to 0$, as $\Theta \to \infty$. □

If we take $t = \frac{1-6\beta}{3}$ and Θ large enough so that $\tau_2 = \frac{t\epsilon}{C_5} - h(\Theta) > 0$, then we have

$$
\begin{aligned}
\lambda^*\left(F_n > \frac{\epsilon n}{C_5}\right) &\le \mathrm{e}^{-t\frac{\epsilon n}{C_5}} E\left(\mathrm{e}^{tF_n}\right), \text{ by Tchebychev's inequality} \\
&\le \mathrm{e}^{-\frac{t\epsilon n}{C_5}} \mathrm{e}^{h(\Theta)n}, \text{ by lemma 8.1} \\
&\le \mathrm{e}^{-\tau_2 n}.
\end{aligned}
$$

Consequently, $\lambda^*\{E_2(n)\} \le \mathrm{e}^{-\tau_2 n}$, for all $n \ge N_2$.

Remark 8.2. Since the growth properties of the space and parameter derivatives along orbits are equivalent (see lemma 4 of [BC85] or lemma 3.4 of [Mo92]), it is possible to build a similar partition on the parameters as Benedicks and Carleson ([BC85, BC91]) did when they built $\mathcal{B}C_1$. Then, using the same kind of arguments as in sections 6 and 8, it is not difficult to bound, on a full Lebesgue measure subset of $\mathcal{B}C_1$, the value of $\frac{C_5}{n}F_n(\xi_0) = \frac{C_5}{n}\sum_{i=1}^{sd_n}\eta_i$, where η_i stands for the depth of the i-th deep essential return of the orbit of ξ_0. This way one obtains the validity of condition (1.2) for the critical point ξ_0, on a full Lebesgue measure subset of $\mathcal{B}C_1$.

9. Uniformness on the choice of the constants

As referred in remark 1.1 all constants involved must not depend on the parameter $a \in \mathcal{B}C_1$. Because there are many constants in question and because they depend on each other in an intricate manner we dedicate this section to clarifying their interdependencies.

We begin by considering the constants appearing in (EG) and (BA) that determine the space $\mathcal{B}C_1$ of parameters. So we fix $c \in [\frac{2}{3}, \log 2]$ and $0 < \alpha < 10^{-3}$.

Then, we consider $\beta > 0$ of definition 2.2 concerning the bound period, to be a small constant such that $\alpha < \beta < 10^{-2}$. A good choice for β would be $\beta = 2\alpha$.

We next fix a sufficiently large Δ such that we have validity on all estimates throughout the text. Most of the times the choice of a large Δ depends on the values of α and β. Note that at no time does the choice of a large Δ depend on the parameter value considered.

After fixing Δ we choose $\frac{2}{3} \le c_0 \le \log 2$ (take, for example, $c_0 = c$), and compute a_0 given by lemma 2.1, and such that (4.3), (4.5), (4.6) and (4.7) hold. Note that this might bring about a reduction in the set of parameters since we will only have to consider parameter values on $\mathcal{B}C_1 \cap [a_0, 2]$ which is still a positive Lebesgue measure set. If necessary we redefine $\mathcal{B}C_1$ to be $\mathcal{B}C_1 \cap [a_0, 2]$.

Finally, we fix any small $\epsilon > 0$ referring to (1.2), and explicit the dependence of the rest of the constants in the table 1

Constant	Dependencies	Main References
B_1	α, β	lemma 2.3
C	α, β	lemma 4.3
d	α, β	(1.1) and (3.2)
τ_1	α	theorem A and section 7
N_1^*	α, Δ, τ_1	(7.2)
N_1	Δ, α, B_1, d, N_1^*	section 3
C_1	N_1, τ_1	theorem A and (3.6)
C_3	α	(5.2)
C_4	β	(5.3)
C_5	α, β	(5.4)
Θ	ϵ, C_5, Δ	sections 5 and 8
γ	Θ	section 2
N_2	Θ	section 8
τ_2	ϵ, C_5, Θ	theorem B and section 8
C_2	N_2, τ_2	theorem B and section 3

TABLE 1. Constants interdependency

In conclusion, all the constants involved depend ultimately on α, β, Δ and ϵ, which were chosen uniformly on $\mathcal{B}C_1$, thus we may say that $(f_a)_{a \in \mathcal{B}C_1}$ is a uniform family in the sense referred to in [Al03].

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