Helping students understand real capacitors: measuring efficiencies in a school laboratory

Paulo Simeão Carvalho[1,2] **and Adriano Sampaio e Sousa**[3]

[1] Departamento de Física da Faculdade de Ciências da Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal
[2] Instituto de Física dos Materiais da Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal
[3] Escola Secundária de Fontes Pereira de Melo, Rua O Primeiro de Janeiro, 4100-366 Porto, Portugal

E-mail: psimeao@fc.up.pt and sampaio.sousa@netcabo.pt

Abstract
A recent reform in the Portuguese secondary school curriculum reintroduced the study of capacitors. Thus we decided to implement some experimental activities on this subject with our undergraduate students in physics education courses. A recent announcement of a new kind of capacitor being developed by a team of scientists at Massachusetts Institute of Technology (MIT), which makes use of nanotechnologies, was a great motivation for the study of a topic that could easily be considered 'out of time'. Since this new kind of capacitor is being seen as the battery of the future, our focus was essentially on efficiency measurements, motivating students to obtain, respectively, the time constant and the energies stored and supplied during the charge and discharge processes, from experimental graphics representing the power as a function of time in real capacitors.

History
The Leiden jar (figure 1) was probably the first capacitor, invented in 1745, in the Netherlands (Greenslade 1994). There are, of course, speculations about its use in ancient times. The Ark of the Covenant, for example, has been described from Exodus to Indiana Jones as having incredibly destructive powers. The meticulous description of its construction that we can read in the Old Testament leads most people to believe that it could have been a giant capacitor. Anyway, the legends underlying the history of the capacitor may be used by teachers as a strong motivation for secondary school pupils.

The recent history of the capacitor has, until now, been less interesting. Its technological applications are well established and known. However, a team of scientists at Massachusetts Institute of Technology, lead by Joel Schindall, has recently announced on Schindall's website[4] something that could really be the battery of the future. It is actually a capacitor whose electrodes are covered with millions of tiny filaments called nanotubes, increasing dramatically its effective area and allowing it to store much more energy. The advantages of this new kind of device are

[4] http://lees-web.mit.edu/lees/schindall_j.htm

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Figure 1. The Leiden jar (courtesy of the Science Museum at the Faculty of Science, University of Porto).

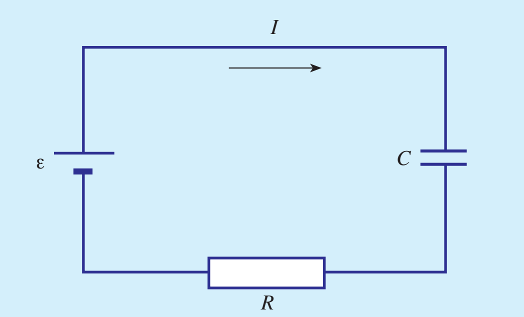

Figure 2. An RC circuit for an ideal capacitor.

many: it has virtually an infinite lifetime, with evident ecological benefits, and can be charged in seconds.

When we read this announcement we felt a mixture of enthusiasm and curiosity. Two questions immediately came to our minds.

(1) The time constant of the capacitive circuit determines the times of both its charge and discharge. How do these scientists manage to keep such a short time of charge and simultaneously provide a very long time of discharge, so that the capacitor can be used as a battery?
(2) A capacitor accumulates energy as an electric field. Will this process be efficient? How can we measure it experimentally?

The first question is hard to answer. The use of ordinary batteries does not depend on the resistance of the external circuit to which they will be connected. If this new device is intended to work in the same way, there will be, of course, some technical issues to be solved that surpass the aim of this work.

The second question can be easily investigated with students, and is discussed in the following.

Theory

When a battery (emf ε and negligible internal resistance) is connected to a series circuit with a resistor (resistance R) and a capacitor (capacitance C) it will act as a source of energy to the capacitor, charging it until the voltage across the capacitor equals the battery emf (figure 2).

For an ideal capacitor (no internal resistance), when the process of charging takes place, the current value decreases and asymptotically approaches zero, as the capacitor becomes charged; meanwhile the voltage increases. This can be described mathematically, making use of Kirchoff's laws for electric circuits, resulting in equations (1) and (2) for the electric current and voltage, respectively:

$$I_c(t) = \frac{\varepsilon}{R} e^{-\frac{t}{RC}} \tag{1}$$

$$V_c(t) = \varepsilon(1 - e^{-\frac{t}{RC}}). \tag{2}$$

The product RC is known as the time constant of the circuit (τ).

If we remove the battery from the circuit and connect the charged capacitor directly to the same resistor R, the voltage and the current follow the same kind of decay curve, as shown by equations (3), and (4), also valid for an ideal capacitor:

$$I_d(t) = -\frac{\varepsilon}{R} e^{-\frac{t}{RC}}. \tag{3}$$

(The negative sign means that the current I_d is flowing in the opposite direction with respect to the charging process.)

$$V_d(t) = \varepsilon e^{-\frac{t}{RC}}. \tag{4}$$

We can easily compute the power received by the capacitor during charge by multiplying equations (1), and (2):

$$P_c(t) = V_c I_c = \frac{\varepsilon^2}{R}(e^{-t/\tau} - e^{-2t/\tau}); \tag{5}$$

and the power supplied by the capacitor during discharge by multiplying equations (3), and (4):

$$P_d(t) = V_d I_d = -\frac{\varepsilon^2}{R} e^{-2t/\tau}, \qquad (6)$$

where the negative sign results from the current direction. Integration of the resultant equations over time yields the corresponding values of electric energy.

Since the voltage V is energy per unit of charge Q, students usually conclude that most probably the energy stored by an ideal capacitor is given by QV. However, it can be shown (Young and Freedman 1996) that half of this value is dissipated in the external resistor regardless of its resistance R, and the other half is stored by the capacitor (Mita and Boufaida 1999, Newburgh 2005). So, students often get surprised that, for an ideal capacitor, the efficiency is just 50%!

In a real capacitor, things may change significantly: the dielectric material between the plates of a real capacitor has a finite resistivity (as compared to infinite resistivity in the case of an ideal capacitor). Therefore, in a real capacitor, a leakage resistance must be included in the capacitor model (Bisquert *et al* 2000). This implies that part of the energy received during charge will not be recovered in discharge: the efficiency will be even lower.

Experimental procedure

In our experiments we studied a 10 F Goldcap electrolytic capacitor (breakdown voltage = 2.3 V) available from Fischer Technik, in a series circuit with a small resistance (<10 Ω) and a variable DC power supply. To acquire data we used current and voltage sensors, connected to a Vernier LabPro interface and LoggerPro software. The data collection rate has an obvious influence on the accuracy of the measurements, so we kept it as high as that supported by the equipment (50 samples per second).

The software was intuitively mastered by the students. They could choose to visualize one or both $V = f(t)$ and $I = f(t)$ plots concerning charge (figure 3(a)) and discharge (figure 3(b)). Then they defined a new variable P as the product of V and I and displayed the $P = f(t)$ plots. Figures 4(a) and (b) show the charge and discharge graphs obtained, respectively.

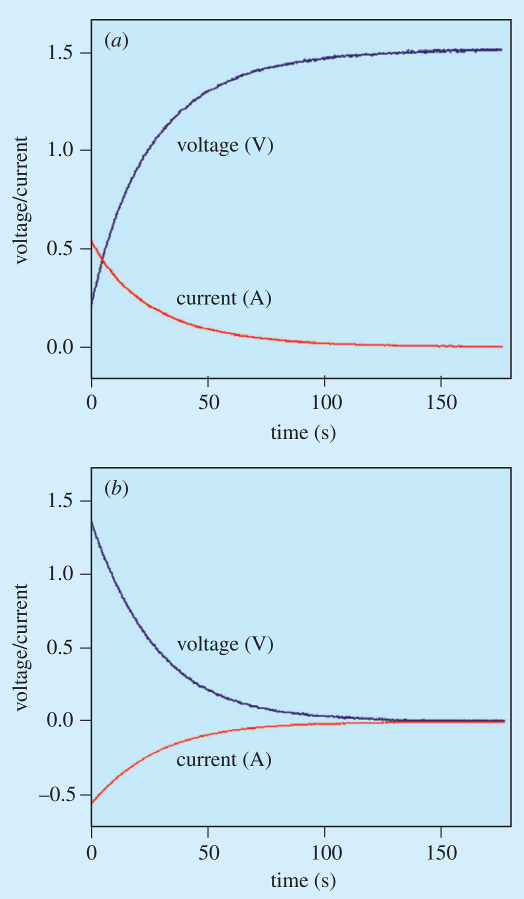

Figure 3. Voltage and current versus time plots concerning: (*a*) charge; and (*b*) discharge.

Results and discussion

The power curve in figure 4(a), concerning charge, reveals a relative maximum at instant t_M. If we look at equation (5), $P_c(t)$ has indeed a predictable maximum, occurring at $t_M = \tau \ln 2$. Therefore, if the data collection rate is high enough, students can get the time constant of the circuit with fair accuracy by reading time t_M in the power graph and making a simple calculation:

$$\tau = \frac{t_M}{\ln 2}. \qquad (7)$$

The power graph for discharge (figure 4(b)) does not exhibit any maximum, and the corresponding function is given by equation (6).

The software also allowed students to determine the values of the underlying areas of these plots, which correspond to the desired values

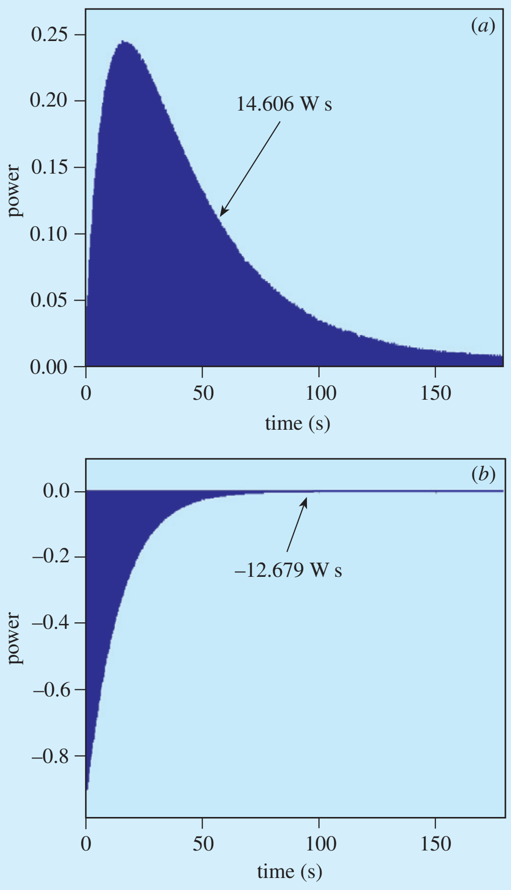

Figure 4. Power versus time plots concerning: (*a*) charge; and (*b*) discharge.

of the energies received (figure 4(a)) and supplied (figure 4(b)) by the capacitor: 14.606 J and 12.679 J, respectively. Contrary to what the students expected, the energy received by the capacitor during charge is *always higher* than the energy yielded on discharge! Where does the energy go, then?

To answer this question, we must recall that we are dealing with a real capacitor and there is going to be a small amount of current flowing between the capacitor plates. We can model a real capacitor by introducing a leakage resistance r, represented by a resistor, connected in parallel with the ideal capacitor, as shown in figure 5.

To determine the power received and supplied by the capacitor, we have to use Kirchoff's laws to find the expressions for $I(t)$, $I_1(t)$, $I_2(t)$ and the voltage V at the capacitor terminals.

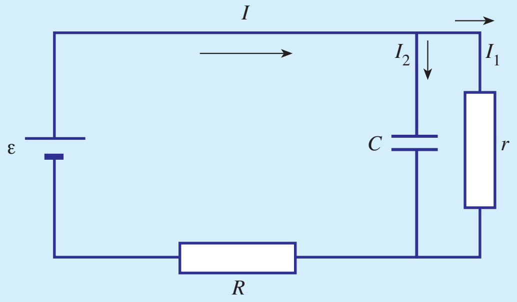

Figure 5. An RC circuit for a real capacitor.

For the process of charge, these expressions are

$$I_{1c}(t) = \frac{q_2}{rC} = \frac{R_{eq}}{rR}\varepsilon(1 - e^{-t/\tau_{eq}}) \tag{8}$$

$$I_{2c}(t) = \frac{\partial q_2}{\partial t} = \frac{\varepsilon}{R}e^{-t/\tau_{eq}} \tag{9}$$

$$I_c(t) = I_{1c}(t) + I_{2c}(t)$$

$$= \frac{\varepsilon}{R}\left[\frac{R_{eq}}{r}(1 - e^{-t/\tau_{eq}}) + e^{-t/\tau_{eq}}\right] \tag{10}$$

$$V_c(t) = \frac{q_2}{C} = \frac{R_{eq}}{R}\varepsilon(1 - e^{-t/\tau_{eq}}) \tag{11}$$

where $R_{eq} = (\frac{1}{r} + \frac{1}{R})^{-1}$, $\tau_{eq} = R_{eq}C$, and q_2 is the electric charge in the capacitor branch.

The power effectively received by the capacitor during charge is given by the product $V_c I_{2c}$. However, what we can experimentally determine is $V_c I_c$:

$$P_c(t) = V_c(t)I_c(t) = \frac{\varepsilon^2}{R^2}R_{eq}\left[\frac{R_{eq}}{r}(1 - e^{-t/\tau_{eq}})^2 + e^{-t/\tau_{eq}}(1 - e^{-t/\tau_{eq}})\right]. \tag{12}$$

P_c has a relative maximum at $t_M = \tau_{eq}\ln\frac{2}{\alpha}$ (see appendix A), where $\alpha = 1 - \frac{R}{r}$. In most cases we will have $\frac{R}{r} \ll 1$, so $\alpha \approx 1$ and therefore the time constant of the circuit is $\tau_{eq} \approx \frac{t_M}{\ln 2}$, which is the same result we had for an ideal capacitor (equation (7)).

The energy E_c received by the capacitor can be computed as follows (see appendix B):

$$E_c = \int_0^{t_c} P_c(t)\,dt \approx \left(\frac{R_{eq}}{R}\right)^2 \frac{C\varepsilon^2}{2}(1 + 2A) \tag{13}$$

with $t_c \gg \tau_{eq}$, where t_c is the instant at which the charge process ends, and $A = \frac{1}{rC}\int_0^{t_c} dt - \frac{3}{2}\frac{R_{eq}}{r}$

corresponds to the fraction of energy dissipated at the leakage resistance r of the real capacitor. It should be emphasized that this term is small *but increases with time*, because current will still flow through the circuit after the capacitor is fully charged; the energy effectively stored by the capacitor during charge is

$$E_{\text{stored}} = \left(\frac{R_{\text{eq}}}{R}\right)^2 \frac{C\varepsilon^2}{2}. \quad (14)$$

For the process of discharge, the analogous expressions to (8), (9), (10), and (11) are

$$I_{1\text{d}}(t) = \frac{q_2}{rC} = \frac{R_{\text{eq}}}{rR}\varepsilon \mathrm{e}^{-t/\tau_{\text{eq}}} \quad (15)$$

$$I_{2\text{d}}(t) = \frac{\partial q_2}{\partial t} = -\frac{\varepsilon}{R}\mathrm{e}^{-t/\tau_{\text{eq}}}. \quad (16)$$

(The negative sign means that the current I_2 in figure 5 is now flowing in the opposite direction.)

$$I_{\text{d}}(t) = I_{1\text{d}}(t) + I_{2\text{d}}(t) = \frac{\varepsilon}{R}\mathrm{e}^{-t/\tau_{\text{eq}}}\left[\frac{R_{\text{eq}}}{r} - 1\right] \quad (17)$$

$$V_{\text{d}}(t) = \frac{q_2}{C} = \frac{R_{\text{eq}}}{R}\varepsilon \mathrm{e}^{-t/\tau_{\text{eq}}}. \quad (18)$$

Once again, the power supplied by the capacitor is $V_{\text{d}} I_{2\text{d}}$, but what we experimentally determine is $V_{\text{d}} I_{\text{d}}$:

$$P_{\text{d}}(t) = V_{\text{d}}(t) I_{\text{d}}(t) = \frac{\varepsilon^2}{R^2} R_{\text{eq}} \mathrm{e}^{-2t/\tau_{\text{eq}}}\left[\frac{R_{\text{eq}}}{r} - 1\right]. \quad (19)$$

This equation shows that the value of P_{d} decreases with time. The corresponding energy supplied by the capacitor to the resistance R is

$$E_{\text{d}} = \int_0^{t_d} P_{\text{d}}(t)\,\mathrm{d}t \approx -\left(\frac{R_{\text{eq}}}{R}\right)^2 \frac{C\varepsilon^2}{2}(1 - B) \quad (20)$$

with $t_d \gg \tau_{\text{eq}}$, where t_{d} is the instant at which the discharge process ends, and $B = \frac{R_{\text{eq}}}{r}$ corresponds to the fraction of energy dissipated at the leakage resistance r of the capacitor. Once again, the negative sign for E_{d} comes from the current $I_{2\text{d}}$.

Comparing equations (13) and (20), we conclude that

$$|E_{\text{c}}| - |E_{\text{d}}| = \left(\frac{R_{\text{eq}}}{R}\right)^2 \frac{C\varepsilon^2}{2}\{|1 + 2A| - |1 - B|\} = \left(\frac{R_{\text{eq}}}{R}\right)^2 C\varepsilon^2 \left(\frac{1}{rC}\int_0^{t_{\text{c}}} \mathrm{d}t - \frac{R_{\text{eq}}}{r}\right)$$

$$|E_{\text{c}}| - |E_{\text{d}}| = \frac{1}{\left(1 + \frac{R}{r}\right)^2} \frac{C\varepsilon^2}{r} \times \left(\frac{\int_0^{t_{\text{c}}} \mathrm{d}t}{C} - \frac{R}{1 + \frac{R}{r}}\right) > 0. \quad (21)$$

Considering that in most cases we have $\frac{R}{r} \ll 1$, then equation (20) can be simplified into

$$|E_{\text{c}}| - |E_{\text{d}}| \approx \frac{\varepsilon^2}{r}\left(\int_0^{t_{\text{c}}} \mathrm{d}t - RC\right). \quad (22)$$

Thus the energy measured when charging the capacitor must effectively be higher than the one measured when discharging it, as experimentally determined (figures 4(a) and (b)); the difference is only due to the energy dissipated at the leakage resistance of the capacitor.

An interesting observation, however, is that the longer the time we take to charge and discharge the capacitor, the higher will be the difference in the measured energies, i.e., the higher will be the difference between the energy supplied to the capacitor and the energy the capacitor really supplies to the discharging circuit!

In this experiment, this difference calculated from figures 4(a) and (b) is $|E_{\text{c}}| - |E_{\text{d}}| = 1.927$ J; also, $\varepsilon \approx 1.6$ V; $C = 10$ F; $R \approx 3\ \Omega$ and the time for charging and discharging was about 180 s. Therefore we are able to compute the effective leakage resistance of the capacitor, by transforming equation (22) into

$$r \approx \frac{\varepsilon^2\left(\int_0^{180} \mathrm{d}t - RC\right)}{|E_{\text{c}}| - |E_{\text{d}}|},$$

obtaining, as a result, $r \approx 199\ \Omega$. Note that, according to this result, $\tau_{\text{eq}} \approx 29.6$ s, which is in good agreement with the assumptions made during the theoretical approach, i.e., $t_{\text{c}}, t_d \gg \tau_{\text{eq}}$ and $\frac{R}{r} \ll 1$.

The results show that the leakage resistance of the real capacitor is not as high as we could expect, but half an hour after its charge it still may have a significant part of its energy stored. That is why we must wait some hours before opening high voltage electronic circuits like TVs or computer monitors, because of the risk of electric shock!

Conclusions

Classical physics topics are generally considered ‘old fashioned’ for classroom discussion. In this work with our students we took advantage of an interesting context and focused unexplored details, in order to motivate them for laboratory work in the shape of a genuine investigation.

The results obtained allowed them to understand the difference between an ideal capacitor and a real one. They used voltage and current sensors to study the charge and discharge of a capacitor, plotted $V = f(t)$, $I = f(t)$ and $P = f(t)$ graphs, and computed the energies received and supplied by the capacitor. These energies were 14.606 J and 12.679 J, respectively; the difference is due to the real behaviour of the capacitor dielectric. These values are, of course, obtained from the measurement of current in the external circuit and do not reflect the exact values of the energies stored in the capacitor. Therefore, students could conclude that only 86.8% of the energy received by the capacitor was returned to the circuit and not 100% as they could expect for an ideal capacitor.

It should be additionally emphasized that the energy received by the capacitor is already just half of the total energy yielded by the power supply. So, the final efficiency of the capacitor is always less than 50%!

Appendix A

The time at which the maximum of P_c occurs can be obtained from the time derivative of equation (12):

$$P_c(t) = V_c(t)I_c(t) = \frac{\varepsilon^2}{R^2}R_{eq} \times \left[\frac{R_{eq}}{r}(1-e^{-t/\tau_{eq}})^2 + e^{-t/\tau_{eq}}(1-e^{-t/\tau_{eq}})\right]$$

by imposing that $\frac{\partial P_c}{\partial t} = 0$. From this derivative, we obtain the following equations:

$$\frac{e^{-t/\tau_{eq}}}{\tau_{eq}}\left\{2\frac{R_{eq}}{r}(1-e^{-t/\tau_{eq}}) - 1 + 2e^{-t/\tau_{eq}}\right\} = 0$$

$$2\frac{R_{eq}}{r} - 1 - 2e^{-t/\tau_{eq}}\left(\frac{R_{eq}}{r} - 1\right) = 0.$$

From this, we take

$$e^{-t/\tau_{eq}} = \frac{2\frac{R_{eq}}{r} - 1}{\frac{R_{eq}}{r} - 1}\frac{1}{2} = \left(1 - \frac{R_{eq}}{r}\right)\frac{1}{2}$$

and therefore, $\frac{-t}{\tau_{eq}} = \ln\frac{\alpha}{2}$, where $\alpha = (1 - \frac{R_{eq}}{r})$, which finally leads to the expression of the relative maximum of P_c at t_M: $t_M = \tau_{eq}\ln\frac{2}{\alpha}$.

Appendix B

The energy E_c received by the capacitor can be computed as $E_c = \int_0^t P_c(t)\,dt$, where t_c is the instant at which the charge process ends. The rigorous calculus of this integral is

$$E_c = \int_0^{t_c} P_c(t)\,dt = \frac{\varepsilon^2}{R^2}R_{eq}\int_0^{t_c}\left\{\frac{R_{eq}}{r}(1-e^{-t/\tau_{eq}})^2 + e^{-t/\tau_{eq}}(1-e^{-t/\tau_{eq}})\right\}dt$$

$$E_c = \int_0^{t_c} P_c(t)\,dt = \frac{\varepsilon^2}{R^2}R_{eq}\left\{\frac{R_{eq}}{r}\int_0^{t_c}(1-e^{-t/\tau_{eq}})^2\,dt + \int_0^{t_c} e^{-t/\tau_{eq}}(1-e^{-t/\tau_{eq}})\,dt\right\}.$$

We can easily solve these integrals in the following way:

$$\int_0^{t_c}(1-e^{-t/\tau_{eq}})^2\,dt = \int_0^{t_c}(1-2e^{-t/\tau_{eq}} + e^{-2t/\tau_{eq}})\,dt = \int_0^{t_c}dt + 2\tau_{eq}\left[e^{-t_c/\tau_{eq}} - 1\right] - \frac{\tau_{eq}}{2}\left[e^{-2t_c/\tau_{eq}} - 1\right].$$

Considering that, experimentally, $t_c \gg \tau_{eq}$, then $e^{-t_c/\tau_{eq}} \approx 0$, and the previous integral becomes

$$\int_0^{t_c}(1-e^{-t/\tau_{eq}})^2\,dt \approx \int_0^{t_c}dt - \tfrac{3}{2}\tau_{eq}.$$

In the same way,

$$\int_0^{t_c} e^{-t/\tau_{eq}}(1-e^{-t/\tau_{eq}})\,dt = \int_0^{t_c} e^{-t/\tau_{eq}}\,dt - \int_0^{t_c} e^{-2t/\tau_{eq}}\,dt = -\tau_{eq}\left[e^{-t_c/\tau_{eq}} - 1\right] + \frac{\tau_{eq}}{2}\left[e^{-2t_c/\tau_{eq}} - 1\right].$$

Once again, considering that, experimentally, $t_c \gg \tau_{eq}$, then $e^{-t_c/\tau_{eq}} \approx 0$, and we obtain

$$\int_0^{t_c} e^{-t/\tau_{eq}}(1-e^{-t/\tau_{eq}})\,dt \approx \frac{\tau_{eq}}{2}.$$

Replacing the integrals in equation (13), we finally compute the energy received by the capacitor during the charge process:

$$E_c = \int_0^{t_c} P_c(t)\,dt \approx \frac{\varepsilon^2}{R^2} R_{eq} \times \left\{ \frac{R_{eq}}{r} \left[\int_0^{t_c} dt - \frac{3}{2}\tau_{eq} \right] + \frac{\tau_{eq}}{2} \right\}.$$

Remembering that $\tau_{eq} = R_{eq}C$, and replacing it in the previous equation,

$$E_c = \int_0^{t_c} P_c(t)\,dt \approx \frac{\varepsilon^2}{R^2} R_{eq} \times \left\{ \frac{R_{eq}C}{rC} \int_0^{t_c} dt - \frac{3}{2}\frac{R_{eq}^2 C}{r} + \frac{R_{eq}C}{2} \right\}$$

$$E_c = \int_0^{t_c} P_c(t)\,dt \approx \left(\frac{R_{eq}}{R}\right)^2 \frac{C\varepsilon^2}{2}(1 + 2A)$$

where $A = \frac{1}{rC}\int_0^{t_c} dt - \frac{3}{2}\frac{R_{eq}}{r}$.

Received 30 November 2007, in final form 26 March 2008
doi:10.1088/0031-9120/43/4/008

References

Bisquert J, Garcia-Belmonte G and Fabregat-Santiago F 2000 The role of instrumentation in the process of modeling real capacitors *IEEE Trans. Educ.* **43** 439–42
Greenslade T B Jr 1994 Discovery of the Leiden jar *Phys. Teach.* **32** 536–7
Mita K and Boufaida M 1999 Ideal capacitor circuits and energy conservation *Am. J. Phys.* **67** 737–9
Newburgh R 2005 Two theorems on dissipative energy losses in capacitor systems *Phys. Educ.* **40** 370–2
Young H D and Freedman R A 1996 *University Physics* 9th edn (New York: Addison-Wesley) p 780

Paulo Simeão Carvalho is an assistant professor at the Porto University Physics Department. His research interests are highly polar liquid crystals, liquid crystal–polymer composites and physics education.

Adriano Sampaio e Sousa is a teaching assistant at the Porto University Physics Department. His research interests are mainly in teaching strategies and conceptual learning. He has worked in teacher training (initial and continuous) for the last 25 years.